SEMINAR ON REAL REDUCTIVE GROUPS AND D-MODULES

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1. Talks

1.1. $\mathrm{SL}_2(\mathbb{C})$**: Forms, Lie Algebras and Weyl's Unitary Trick.** (Benedek Dombos, 27.10.20) This talk should make us comfortable with our main example SL_2 .

- Discuss all the different incarnations of $\mathrm{SL}_2(R)$: on can see it as an algebraic group, a topological group, a Lie group or an abstract group (also depending on the ring R). Discuss $\mathfrak{sl}_2$.
- Reductive groups and Lie algebras admit interesting substructures and decompositions. Explain some of them on the example of SL_2. In particular the Cartan, Bruhat, Iwasawa and Jordan-Chevalley decomposition.
- The second half of your talk should concern representation theory. Distinguish the different meanings of representations: algebraic, analytic, smooth, etc.
- Explain Weyl's unitary trick and how to identify finite dimensional
  - representations of $\mathfrak{sl}_2(\mathbb{C})$,
  - smooth representations of $\mathrm{SL}_2(\mathbb{R})$,
  - algebraic/holomorphic representations of $\mathrm{SL}_2(\mathbb{C})$ and
  - unitary representations of $\mathrm{SU}(2)$.
- Classify the finite dimensional representations of $\mathfrak{sl}_2(\mathbb{C})$. Be sure do draw them as diagrams.
- As an application, mention the Lefschetz decomposition of the cohomology of a Kähler manifold.
- If you have time left explain the finite dimensional *smooth* representations of $\mathrm{SL}_2(\mathbb{C})$.

References: are abundant. A nice list of decompositions can be found here. For the unitary trick see [Kna01].

1.2. $\mathrm{SL}_2(\mathbb{R})$**: Infinitesimal Equivalence, Harish-Chandra modules and Unitary Representations.** (Hannes Kristinn Árnason, 3.11.20) We start our discussion of *infinite-dimensional* representations of real reductive groups on the example of $\mathrm{SL}_{2(\mathbb{R})}$.

- Explain different notions of infinite dimensional representation: An action on a Banach space, Hilbert space, a locally convex topological vector space, etc.
- Explain the following dilemma. By Gelfand's philosophy, one should study a space X with the action of a group G by studying the functions on X. However, different choices of function space give non-equivalent representations.

- Define the notion of infinitesimal equivalence and Harish-Chandra modules, and explain how this can circumvent the dilemma. Use the example of $S^1 = \mathrm{SO}(2)$ and Fourier series/polynomials.
- Classify the irreducible admissible $(\mathfrak{sl}_2(\mathbb{R}), \mathrm{SO}(2))$-modules and discuss which of them are unitary. Use diagrams instead of formulas whenever possible.

Reference: [Cas] is a really nice essay. If you do the classification of irreducible $(\mathfrak{g}, K)$-modules from Section 3, please draw the diagrams from Section 8. For the classification of the unitary representations see Chapter III in [HT92]. Another source is [Ker14].

1.3. $\mathrm{SL}_2(\mathbb{R})$**: Principal Series, Discrete Series and Modular Forms.** (Xiaoxiang Zhou, 10.11.20) We find function space interpretations for the Harish-Chandra modules classified above.

- Define the principal series as the space $\mathrm{Ind}^\infty(\chi)$ for a character $\chi : B \to \mathbb{C}^\star$, for $B \subset \mathrm{SL}_2(\mathbb{R})$ the group of upper triangular matrices.
- Explain how $\mathrm{Ind}^\infty(\chi)$ is dual to $\mathrm{Ind}^\infty(\chi^{-1})$.
- Explain how $\mathfrak{sl}_2(\mathbb{R})$, $\mathrm{SO}(2)$ and the Casimir element acts.
- Discuss when they are reducible or irreducible, and how all the Harish-Chandra modules from the last talk appear as subquotients.
- Discuss which of them are unitary.
- Draw a picture visualizing the parameters in $\mathbb{C}$ (purely imaginary line and real unit interval).
- Discuss the discrete series representations. Try to answer the following questions:
  - How can we realize the discrete series in terms of functions on the group or sections in line bundles on the upper half plane $\mathcal{H}$.
  - What is the holomorphic and antiholomorphic discrete series. Make sense of the translation *holomorphic⇄highest weight vector.*
- Discuss the relation to modular forms:
  - Recall the definition of a modular form.
  - Explain the relation to functions on the double quotient
$$\mathrm{SL}_2(\mathbb{Z})\backslash \mathrm{SL}_2(\mathbb{R})/ \mathrm{SO}(2).$$
  - It would be desirable to understand a sentence of the form "A modular form (cusp form of weight ≥ 2) is a highest weight vector of a discrete series summand of $L^2(\mathrm{SL}_2(\mathbb{Z})\backslash \mathrm{SL}_2(\mathbb{R}))$".

This talk is closely related to the preceding one. Make sure to coordinate your talks. Reference: Again Casselmann's essay [Cas]. In particular: Section 8, 19,20,21. [Kob93] for modular forms and elliptic curves. For the connection to representation theory [Bum97], and for a nice informal account, just concerned with modular forms, see here[Ven].

1.4. **Real Reductive Groups: Definitions, Involutions and Decompositions.** (Berber Lorke, 17.11.20) The goal is to extend the discussion in Talk 1 to arbitrary real reductive groups.

- Recall the definition of a semisimple/reductive complex linear algebraic group.
- Define real semisimple/reductive Lie algebras/groups. Note that there are various definitions of real reductive groups. Discuss the classical groups.

- Explain the Cartan decomposition, compact and split forms.
- Quickly discuss Satake diagrams.
- Explain the Bruhat, Iwasawa and Jordan-Chevalley decompositions. Use the example GL_n and explain the connection to theorems in linear algebra.

If you are overwhelmed by the content either use a simple definition of real reductive group or restrict to semisimple groups. Reference: You can use [Kna01] which provides many examples. Another source is [Wal88]. Also take a look at Chapter 24 in [Bor91].

1.5. **Real Reductive Groups: Harish-Chandra Modules and Langlands Classification.** (Rostislav Devyatov, 24.11.20)

1.6. **Complex Reductive Groups: Harish-Chandra Bimodules and Category $\mathcal{O}$.** (Liao Wang, 1.12.20)

1.7. **D-Modules: Linear PDE's, Weyl Algebras and Symbols.** (Thibaud van den Hove, 8.12.20)

1.8. **D-Modules: Smooth Varieties, Coordinates and Defintions.** (Jiexiang Huang, 15.12.20)

1.9. **D-Modules: Functorialities and Kashiwara's Theorem.** (Timm Peerenboom, 22.12.20)

1.10. $\mathbb{P}^1$**: Line bundles, Representations and Grothendieck Resolution.** (Daniel Bermudez, 12.1.21)

1.11. $\mathbb{P}^1$**: Global Differential Operators, Global Sections and the Localisation Theorem.** (Jon Miles, 19.1.21)

1.12. $\mathbb{P}^1$**: Principal Series and Discrete Series via D-Modules.** (Konrad Zou, 26.1.21)

1.13. **Flag varieties: Localisation Theorem.** (Mingyu Ni, 2.2.21)

1.14. **Flag varieties: K-orbits, Local Systems and Langlands Classification.** (9.2.21)

References

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[Cas] Bill Casselmann, *Admissible representations of* $\mathrm{SL}(2,\mathbb{R})$, Essay on `https://www.math.ubc.ca/~cass/research/essays.html`.

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[Ker14] Matt Kerr, *Notes on the representation theory of sl2 (r)*, Hodge Theory, Complex Geometry, and Representation Theory **608** (2014), 173.

[Kna01] Anthony W. Knapp, *Representation theory of semisimple groups*, Princeton Landmarks in Mathematics, Princeton University Press, Princeton, NJ, 2001, An overview based on examples, Reprint of the 1986 original. MR 1880691

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[Wal88] Nolan R Wallach, *Real reductive groups i*, Academic press, 1988.