SEMINAR: HALL ALGEBRAS AND QUANTUM GROUPS WS2019

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1. Talks

1.1. **Talk 1: Hall Algebras.** (10.10.19; Busse, Hanna) The goal of this talk is to introduce the natural habitat and definition of Hall algebras, following [Sch06, Section 1.1-1.3]

- Define a *finitary* abelian category. Recall the Grothendieck group of an abelian category, explain the Grothendieck group in finite length categories (basis of simples). Recall global dimension. [Sch06, Section 1.1]
- Give examples of such categories, explain which properties they have. Those examples will come back in later talks, and will be properly introduced there. So just roughly explain.
  - Quiver representations. Explain how the standard resolution of the simples shows that this category has global dimension 1.
  - Nilpotent representation of the Jordan quiver.
  - Coherent sheaves on $\mathbb{P}^1$.
- Define the Euler form. Explain that it descends to the Grothendieck group. [Sch06, Section 1.2]
- Explain the definition of the Hall algebra, [Sch06, Start of Section 1.3]. Do not give the function definition.
- Prove that the Hall algebra is associative. [Sch06, Proposition 1.1]
- Explain how the Hall algebra is graded by the Grothendieck group. [Sch06, Remark 1.3]

As a running example, compute everything explicitly for $\mathbf{H}_{\mathrm{Vect}_{\mathbb{F}_q}}$, the Hall algebra of the category of finite dimensional vector spaces over a finite field, see [Sch06, Example 3.14].

1.2. **Talk 2: Green's Coproduct.** (17.10.19; De Vries, Sjoerd) In this talk, we want to understand how one can (attempt) to upgrade the Hall algebra to a *bialgebra,* following [Sch06, Sections 1.4, 1.5].

- Define the completed tensor product $\mathbf{H}_{\mathcal{A}}\widehat{\otimes}\mathbf{H}_{\mathcal{A}}$. [Sch06, Section 1.4]
- Define the coproduct. Show that the coproduct is coassociative. [Sch06, Proposition 1.4]
- Compute products and coproducts in the examples $\mathbf{H}_{\mathrm{Vect}_{\mathbb{F}_q}}$, and $\mathbf{H}_{\mathrm{Rep}_{\mathbb{F}_q}(A_2)}$.
- Mention all the points in [Sch06, Remark 1.6], give an example for the failure of (co)commutativity.
- Prove [Sch06, Lemma 1.7].
- Explain the twisted multiplication. [Sch06, Section 1.5]
- Prove [Sch06, Lemma 1.8].

- State but *do not prove* Green's theorem. Explain where the assumption of global dimension ≤ 1 is used. [Sch06, Theorem 1.9]
- Explain why the twisted multiplication is necessary using the example $\mathbf{H}_{\mathrm{Vect}_{\mathbb{F}_q}}$.
- Define the extended Hall algebra and state [Sch06, Corollary 1.10].

1.3. **Talk 3:** $\dot{\mathrm{U}}_q(\mathfrak{sl}_2)$**.** (24.10.19; Lorke, Berthold) In this talk we want to understand how one could get from $\mathfrak{sl}_2$ to the definition of the quantum group $\mathrm{U}_v(\mathfrak{sl}_2)$ and then study some properties and structures of $\mathrm{U}_v(\mathfrak{sl}_2)$.

- Recall generators and relations of $\mathfrak{sl}_2$ and $\mathrm{U}(\mathfrak{sl}_2)$. [Lau11, Start of Section 1.2]
- Recall the finite dimensional representations of $\mathfrak{sl}_2$. Draw the usual pictures of the irreducible representations, using the basis $F^k/k!v$ for a highest weight vector v, annotate the arrows with the coefficients with respect to this basis.
- Define $\dot{\mathrm{U}}(\mathfrak{sl}_2)$ as in the end of [Lau11, Section 1.2], but for $\dot{\mathrm{U}}(\mathfrak{sl}_2)$ instead of $\dot{\mathrm{U}}_q(\mathfrak{sl}_2)$.
- Define the quantum integers, factorials.
- Change the integers from the pictures to quantum integers.
- Observe how the relation $EF - FE$ changes.
- Define $\dot{\mathrm{U}}_q(\mathfrak{sl}_2)$. [Lau11, End of Section 1.2]
- Define $\mathrm{U}_v(\mathfrak{sl}_2)$. [Lau11, Middle of Section 1.2]
- Prove that one can turn $\mathrm{U}_v(\mathfrak{sl}_2)$ into a Hopf algebra. [Jan96, Lemma 3.1, 3.4, 3.6, 3.7 and 3.8]
- Show that $\mathrm{U}_v(\mathfrak{sl}_2)$ is neither commutative nor cocommutative. Compare this to the $\mathrm{U}(\mathfrak{sl}_2)$, which is cocommutative.
- Explain quickly that we can now take tensor products of representations of $\mathrm{U}_v(\mathfrak{sl}_2)$.
- Let V be the fundamental (two-dimensional) representation of $\mathrm{U}_v(\mathfrak{sl}_2)$. Show that the map swapping the factors in $V \otimes V$ does not commute with the action of $\mathrm{U}_v(\mathfrak{sl}_2)$. Explain how the swap map has to be modified in this example. Write down explicit matrices. [Jan96, 3.14, 3.15] Connect this with the non-cocommutativity.

1.4. **Talk 4: A Geometric Constructions of Representations of** $\mathrm{U}_v(\mathfrak{sl}_2)$ **and Definition of** $\mathrm{U}_v(\mathfrak{g})$**.** (31.10.19; Liao, Wang) This talk has two goals. First, we want to understand a geometric construction of finite dimensional irreducible representations of $\mathrm{U}_v(\mathfrak{sl}_2)$. Then, we want to see the general definition of $\mathrm{U}_v(\mathfrak{g})$ and attempt to understand some of its relations.

- Define the varieties $\mathfrak{M}(w,d), \mathfrak{M}(d), \mathfrak{M}(w, w+1, d)$ and the correspondence $\mathfrak{M}(w,d) \leftarrow \mathfrak{M}(w, w+1, d) \to \mathfrak{M}(w, d+1)$.[Sav03, Section 1.3]
- Explain how a correspondence induces a linear map between the function spaces.[Sav03, Section 1.3]
- Define the action of $\mathrm{U}_v(\mathfrak{sl}_2)$ on the function spaces.
- Prove that the action is well-defined and the resulting representations coincides with irreducibles from last lecture. [Sav03, Proposition 1.3.1]
- Introduce the tensor product variety $\mathcal{T}(d)$. Focus on the case of tensor products of two irreducibles. Explain how $\mathrm{U}_v(\mathfrak{sl}_2)$ acts. *Do not* prove [Sav03, Theorem 2.1]. [Sav03, Section 2.1, 2.2]

- Now we define $\mathrm{U}_v(\mathfrak{g})$ for a semisimple Lie algebra $\mathfrak{g}$ as in [Jan96, Chapter 4]
  - Quickly mention that a semisimple Lie algebra comes with a root system, Cartan matrix, etc., [Jan96, 4.1] and that one can write down generators and relations for $\mathfrak{g}$ using this data [Jan96, Introduction Chapter 4]. Make everything as explicit as possible in the example $\mathrm{A}_2 = \mathfrak{sl}_3$.
  - Define $\mathrm{U}_v(\mathfrak{g})$ by first giving the "$\mathfrak{sl}_2$" relations (R1)-(R4) and then the Serre relations (R5) and (R6). [Jan96, 4.3] Again, explain the relations for $\mathrm{A}_2 = \mathfrak{sl}_3$.
  - Define $\mathrm{U}_v(\mathfrak{n}^+)$ and $\mathrm{U}_v(\mathfrak{b})$ (Jantzen calls $\mathrm{U}_v(\mathfrak{n}^+) = \mathrm{U}_v(\mathfrak{g}^+)$). [Jan96, 4.4].

1.5. **Talk 5: Quiver Representations.** (7.11.19; Seifner, Patrick) A big class of examples of interesting Hall algebras comes from categories of representations of quivers over a finite field. In this talk we collect facts about quiver representations and witness the first signs of the Ringel–Green Theorem.

- Define what a quiver and a quiver representation really is. [Sch06, Introduction Section 3.1]
- State important properties of $\mathrm{Rep}_k^{nil}(Q)$: abelian, Krull–Schmidt, classification of simples, dimension vector/description of Grothendieck group, computation of Euler form. [Sch06, Introduction Section 3.1, Proposition 3.1, Corollary 3.2, Proposition 3.3]
- Explain [Sch06, Corollary 3.1]. No need to use the word Kac–Moody.
- Without proof, how the quivers of finite type look like (ADE), [Sch06, Appendix A.1]. State Gabriel's Theorem, see [Sch06, Theorem 3.7]. Emphasize that everything is independent of the field.
- Go through [Sch06, Example 3.8] with great care and detail. Make everything explicit for $n = 2$.
- Do [Sch06, Example 3.9].
- Observe that the graded dimensions of $\mathrm{gr}\,\mathrm{U}(\mathfrak{n}^+)$ and the Hall algebra coincide by Gabriel's Theorem. This is a first hint towards the Ringel–Green Theorem.
- Do the computation in [Sch06, Point 2 and 3, Example 3.15]. Send a cold shiver down our spines when we encounter the Serre relations for $\mathrm{A}_2 = \mathfrak{sl}_3$. Emphasize that all relations are polynomial in q.
- State the Ringel–Green Theorem [Sch06, Theorem 3.16] in the ADE case.

1.6. **Talk 6: Ringel–Green Theorem.** (14.11.19; Kornemann, Malte und Wojciechowski, Zbigniew) In this talk we want to prove three quarter (the well-definedness and injectivity part) of the Ringel–Green Theorem:

$$\Psi_\nu : \mathrm{U}_\nu(\mathfrak{n}_Q^+) \hookrightarrow \mathbf{H}_Q.$$

We consider more general the case an arbitrary Quiver without loops (not only ADE as in the last talk).

- Define $\mathrm{U}_\nu(\mathfrak{b}_Q)$ and $\mathrm{U}_\nu(\mathfrak{n}_Q^+)$ associated to a quiver without loops by generators and relations. [Sch06, Above Theorem 3.16]. Explain that these are *specializations* at $\nu = q^{\frac{1}{2}}$ of $\mathrm{U}_v(\mathfrak{b}_Q)$ and $\mathrm{U}_v(\mathfrak{n}_Q^+)$, and q is the cardinality of

our finite field. Remark that the definition coincides with the Lie algebra definition if Q is an Dynkin quiver.
- Again, state the Ringel–Green Theorem. [Sch06, Above Theorem 3.16]
- We prove that the map is well-defined and injective:
  - Show that Hall algebra fulfills the necessary relations. Explain that in the case $|a_{i,j}| \leq 1$ this is just the calculation from the last talk, see [Sch06, Point 2 and 3, Example 3.15]. Prove the general case as in [Sch06, Proof of Theorem 3.16]. This gives us a well-defined map.
  - Show injectivity by using Green's Scalar product. State [Sch06, Proposition 1.12, Corollary 1.13]. Use Green's Scalar product to show injectivity as in [Sch06, Proof of Theorem 3.16].
- Remark that we can prove surjectivity if we now that the graded dimensions agree. Recall how Ringel's theorem gives us the graded dimension of of $\mathbf{H}_Q$. Compare this to the formula graded dimension of $\mathrm{U}_\nu(\mathfrak{n}_Q^+)$, which we will prove in the next talk by providing a PBW style theorem. [Sch06, Proof of Theorem 3.16]
- Explain that the statement can be upgraded to $\mathrm{U}_\nu(\mathfrak{b}_Q)$ and the extended Hall algebra.
- Now we want to see that the structure constants of the $\mathbf{H}_Q$ for quivers of finite type are polynomials in q, and that one can define a generic Hall algebra with those polynomials as structure constants, (then $\mathbf{H}_Q$ is obtained by specialization, in same way as $\mathrm{U}_\nu(\mathfrak{n}_Q^+)$). For this, state and not prove [Sch06, Proposition 3.18] and do [Sch06, Example 3.20, 3.21].
- For quivers which are not of finite type, the surjectivity of Ψ_ν fails. We want to see this in the example of the Kronecker quiver $Q = \bullet \rightrightarrows \bullet$.
  - Remark that the PBW theorem holds here as well and would give us $\dim \mathrm{U}_v(\mathfrak{n}_+)_{(1,1)} = 2$.
  - Classify the representations of Q over $\mathbb{F}_q$ with dimension vector $(1,1)$ and conclude that $\dim(\mathbf{H}_Q)_{(1,1)} = 2 + q$.
  - Conclude that Ψ_ν is not surjective.

1.7. **Talk 7: PBW Bases and $U_q(\mathfrak{g})$.** (21.11.19; Bonfert, Lukas)

The goal of this talk is to prove an analogue of the Poincaré–Birkhoff–Witt theorem for $U_q(\mathfrak{g})$, thereby completing the proof of the Ringel–Green theorem.

- Recall the Weyl group W and braid group B associated to a finite type root system R. Recall the length function $l : W \to \mathbb{Z}_{\geq 0}$.
- State, without proofs, the results about braid group actions in the classical (non-quantized) case. The main result is [DDPW11, Theorem 5.26]. A general overview is contained in [DDPW11, Sections 5.4-5].
- Define a group homomorphism $T : B \to \mathrm{Aut}(U_v(\mathfrak{g}))$ [Lus90, §1.3]. (Note: Lusztig's $U_{\mathcal{A}'}$ is our $U_v(\mathfrak{g})$ while his U is the integral form of $U_v(\mathfrak{g})$.)
- Prove [Lus90, Proposition 1.7].
- Construct a putative basis of $U_v(\mathfrak{g})$ using the braid group action. Prove that it is indeed a linearly independent subset [Lus90, Proposition 1.10]. The proof relies on a [Lus88, Theorem 4.12]; state this result, as it applies to the present setting, but do not prove it.
- Exhibit a triangular decomposition of $U_v(\mathfrak{g})$; see [Ros88, Proposition 2].
- Prove the PBW theorem [Lus90, Proposition 1.13].

- Recall how the PBW and Krull–Schmidt theorems imply that, in finite type, $\mathbf{H}_{\mathrm{Rep}_{\mathbb{F}_q}(Q)}$ and $U_{\sqrt{q}}(\mathfrak{n}_Q^+)$ have the same graded dimension. Conclude the Ringel–Green theorem.

1.8. **Talk 8: Drinfeld Double.** (28.11.19; Spellmann, Jan) In this talk we discuss the Drinfeld double associated to a Hopf algebra and prove that the Drinfeld center of the category of representations of a Hopf algebra coincides with the category of representations of its Drinfeld double. We use this to construct all of $\mathrm{U}_v(\mathfrak{g})$ using Hall algebras.

- Collect a list of all the ingredients that make a Hopf algebra. For example: [Kas95, Chapter III]
- Define the Drinfeld double (also called quantum double) of a Hopf algebra. [Kas95, Definition IX.4.1]
- Sketch a proof why the Drinfeld double is a Hopf algebra. [Kas95, Chapter IX]
- Explain what a tensor category is. [Kas95, Definition XI.2.1]
- Explain what a braided tensor category is. [Kas95, XIII.1.1 Definitions and main properties]
- Define the Drinfeld center of a tensor category. [Xia97, 2.2, 2.3]
- Show that the center is a braided monoidal category. [Kas95, Theorem XIII.4.2.]
- Show [Kas95, Theorem XIII.5.1]!
- Now explain how we can construct $U_q(\mathfrak{g})$ as a Drinfeld double (modulo some simple relations) of a Hall algebra. [Sch06, Section 5.2]

1.9. **Talk 9: "Fun" with $\mathbb{F}_1$.** (5.12.19; Wehrhan, Till) So far, we have constructed $\mathrm{U}_v(\mathfrak{g})$. What about $\mathrm{U}(\mathfrak{g})$? In this talk we will construct $\mathrm{U}(\mathfrak{g})$ using Hall algebras of "quiver representations over $\mathbb{F}_1$".

- Explain that while there is no field $\mathbb{F}_1$, there is often an agreement what certain objects associated to fields "should be" over $\mathbb{F}_1$. [Szc10, Introduction]
- How formulas which describe number of subspaces become formulas for subsets in the limit $q \to 1$. [Szc10, Introduction]
- Define the category $\mathrm{Vect}(\mathbb{F}_1)$ of pointed sets with partial bijections as morphisms. Explain that one can think about the pointed element in the set as the zero vector and the other elements as basis vectors. Partial bijections can be thought of as linear maps which are adapted to the bases of domain and codomain. [Szc10, Definition 2.1]
- Explain the important properties of $\mathrm{Vect}(\mathbb{F}_1)$ you will need later. [Szc10, List after Definition 2.1]
- Explain the Jordan normal form for $\mathrm{Vect}(\mathbb{F}_1)$. [Szc10, Section 3]
- *Quickly* define quiver representations over $\mathrm{Vect}(\mathbb{F}_1)$. Explain how we can define exact sequences in $\mathrm{Rep}_{\mathbb{F}_1}(Q)$. [Szc10, Section 4]
- Mention without proof and all the details that $\mathrm{Rep}_{\mathbb{F}_1}(Q)$ fulfills the Jordan–Hoelder and Krull–Schmidt theorem.
- Describe the Grothendieck group in terms of dimension vectors. [Szc10, Section 4]
- *Quickly* define the Hall algebra with its multiplication, comultiplication (this is not the Hall coproduct), and state that $\mathbf{H}(Q) = \mathrm{U}(\mathfrak{n}_Q)$, where

$\mathfrak{n}_Q = \mathrm{P}(\mathbf{H}(Q))$ denotes the Lie algebra primitive elements in the Hopf algebra $\mathbf{H}(Q)$. Explain how this is implied by the Milnor–Moore Theorem. [Szc10, Section 6]
- Extend this to define the extended Hall algebra [Szc10, Section 6.1]
- State and prove [Szc10, Theorem 6]. If you need to, say if this has not been done in the talks before, explain how we can associate a Lie algebra to Q before this.
- Explain that $\rho : \mathrm{U}(\mathfrak{b}) \to \widetilde{\mathbf{H}}(Q)$ is determined by a map of Lie algebras, [Szc10, Remark 3]. Mention [Szc10, Example 8.1]
- Prove that ρ is an isomorphism in type A. [Szc10, Section 10]

1.10. **Talk 10: Jordan Quiver I.** (12.12.19; Stelnzer, Jendrik) In this talk we allow loops in our quiver for the first time. In fact, just one loop. We want to understand the Hall algebra associated to representations of the Jordan quiver $\circlearrowleft$.

- Explain that representations of the Jordan quiver are nothing other than vector spaces with an endomorphism. [Sch06, Section 2.1]
- Explain how the Jordan normal form tells us everything about $\mathrm{Rep}_k^{nil}(Q)$. [Sch06, Theorem 2.1]
- Explain that the Hall algebra has a basis given by partitions. [Sch06, After Theorem 2.1]
- Compute some structure constants as in [Sch06, Example 2.2 and 2.3].
- Do the same over $\mathbb{F}_1$. [Szc10, Section 9]
- Define the ring of symmetric functions Λ. [Mac98, Chapter 1, Section 1]
- Explain the various bases: monomial, elementary, complete symmetric functions, power sums, [Mac98, Chapter 1, Sections 2-5]. Emphasize that they are also indexed by partitions.
- Prove that the $\mathbf{H}_{\mathbb{F}_1} \cong \Lambda$. [Mac98, Section 9]

1.11. **Talk 11: Jordan Quiver II.** (19.12.19; Antor, Jonas) We continue our study of the Jordan quiver. This time over $\mathbb{F}_q$, which makes it significantly more interesting. We compute more structure constants and show that they are polynomial in q.

- Do [Sch06, Example 2.4]. A more involved computation of structure constants.
- Prove [Sch06, Theorem 2.6]
- Show the existence of Hall polynomials, and that they specialize to the structure constants of the Hall algebra. [Sch06, Proposition 2.7]
- Define the generic Hall algebra. [Sch06, After Proposition 2.7]
- Compute Green's Scalar product. [Sch06, Lemma 2.8]
- Show that the generic Hall algebra is isomorphic to the ring of symmetric function $\Lambda \otimes \mathbb{C}[t, t^{-1}]$.

1.12. **Talk 12: Cyclic Quiver.** (9.01.20; Akynbaev, Ulukbek) We study the Hall algebra of the cyclic quiver Q. The goal is to understand how the Hall algebra can be decomposed into a tensor product of its *composition subalgebra* and the ring of symmetric functions

- Classify the representations. Draw pictures to illustrate the indecomposables. [Sch06, Beginning of Section 3.5]

- Define the *composition algebra* of a Hall algebra, as the algebra generated by simples. Explain that it is the isomorphic image of the map $\Psi : \mathrm{U}_\nu(\mathfrak{n}_Q^+) \to \mathbf{H}(Q)$.
- Explain and prove [Hub09, Theorem 17].

1.13. **Talk 13:** $\mathrm{U}_v(\widehat{\mathfrak{sl}}_n)$**.** (16.01.20; Pfeifer, Calvin) In this talk we want to understand the affine Lie algebra $\widehat{\mathfrak{sl}}_n$, its q-version $\mathrm{U}_v(\widehat{\mathfrak{sl}}_n)$ and their relation to the Hall algebra of the cyclic quiver.

- Define the loop algebra and affine Lie algebra associated to an affine Dynkin diagram. [Sch06, Example A.15]
- Draw the root system of $\widehat{\mathfrak{sl}}_2$. [Sch06, Example A.15]
- Introduce $\mathrm{U}_v(\widehat{\mathfrak{sl}}_n)$ [Sch06, Appendix A.4]
- Provide the Drinfeld–Jimbo and Drinfeld's new presentation. [Hub19, Section 2.1, 2.2]
- By referring to the last talk, note that $\mathrm{U}_v^{\geq 0}(\widehat{\mathfrak{sl}}_n)$ is isomorphic to the composition algebra of the cyclic Quiver.[Hub19, Theorem 5]
- Provide the different presentations of $\mathrm{U}_v(\widehat{\mathfrak{gl}}_n)$. [Hub19, Section 2.1, 2.2]
- State the result that $\mathrm{U}_v^{\geq 0}(\widehat{\mathfrak{gl}}_n)$ gives all us all of the Hall algebra. Sketch the proof. [Hub19, Main Theorem]

1.14. **Talk 14:** $\mathrm{Coh}(\mathbb{P}^1)$ **I.** (23.01.20; Zhang, Mingjia? and Puhlmann, Luise?) In this talk we want to understand the category $\mathrm{Coh}(\mathbb{P}^1)$ in great detail. We want to classify the indecomposable objects and their extensions.

- Recall the standard affine chart of $\mathbb{P}^1$ with the coordinate rings and transition function. [BK01, Section 2.1]
- Define the category $\mathrm{Coh}(\mathbb{P}^1)$ in elementary terms. [BK01, Section 2.1]
- Define the sheaves $\mathcal{O}(n)$ and compute the Hom spaces between them.
- Define skyscraper sheaves and their locally free resolution.
- Prove [BK01, Proposition 3].
- Compute the Hom and Ext groups between all indecomposables.
- Conclude that $\mathrm{Coh}(\mathbb{P}^1)$ (over a finite field) satisfies the properties needed to define the Hall algebra, and that the subcategory of torsion sheaves induces a subalgebra.

1.15. **Talk 15:** $\mathrm{Coh}(\mathbb{P}^1)$ **II.** (30.01.20; Robin, Louis) In this talk we want to study the Hall algebra of $\mathrm{Coh}(\mathbb{P}^1)$ study its connection with $\mathrm{U}_v(\widehat{\mathfrak{sl}}_2)$

- Compute some Hall numbers. [BK01, Theorem 13]
- Explain the structure of the torsion part of the Hall algebra. [BK01, Proposition 15]
- Explain the tensor decomposition. [BK01, Proposition 20]
- Prove [BK01, Theorem 26]

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