1] Let V be a vector space over a field F and $U, W \subset V$ be subspaces. V is the direct sum of U and W (written $V = U \oplus W$), if

(1) $V = U + W$

(2) $\{0\} = U \cap W$.

Show:

(a) Let β_1, β_2 be bases of U, W, respectively, then $\beta = \beta_1 \cup \beta_2$ is a basis of V if and only if $V = U \oplus W$.

(b) Recall that for two vector spaces V_1, V_2 over F,

$$V_1 \times V_2 = \{(v_1, v_2) \mid v_1 \in V_1, v_2 \in V_2\},$$

the cartesian or direct product of V_1 and V_2 is a vector space over F with componentwise addition/scalar multiplication. Then

$$T: U \times W \to V, \quad (u, w) \mapsto u + w$$

is a linear map. Show that

$$T \text{ is an isomorphism} \iff V = U \oplus W$$

2] Let X, Y be sets, F be a field and $f: X \to Y$ a function. Show that

$$f^*: \mathrm{Fun}(Y, F) \to \mathrm{Fun}(X, F), \quad \alpha \mapsto \left[\alpha f: X \to F,\ x \mapsto \alpha(f(x))\right]$$

is a linear map. One calls f^* the pullback along f.

3) Let X, Y be sets, $f : X \to Y$ be a function. Recall the equivalence relation on X

$$a \sim_f b \iff f(a) = f(b), \text{ for } a, b \in X.$$

Let $[x]_f := \{a \in X \mid x \sim_f a\}$ denote the equivalence class of $x \in X$.

(a) Show that f is injective if and only if $|[x]_f| = 1$ for all $x \in X$.

(b) For $y \in Y$, one calls $f^{-1}(\{y\}) = \{x \in X \mid f(x) = y\}$ the <u>fiber</u> of f over y. Show

$$[x]_f = f^{-1}(\{f(x)\}), \text{ for all } x \in X.$$

(c) Assume that X is finite. Show

$$|X| = \sum_{y \in \mathrm{im}(f)} f^{-1}(\{y\}).$$

4) Let V, W be finite dimensional vector spaces over a field F with bases β and γ. $n = \dim V$, $m = \dim W$. Recall the map

$$\Phi_\beta^\gamma : \mathrm{Hom}(V,W) \longrightarrow F^{m \times n}, \quad T \mapsto [T]_\beta^\gamma .$$

Show that the inverse of Φ_β^γ is given by

$$(\Phi_\beta^\gamma)^{-1} : F^{m \times n} \longrightarrow \mathrm{Hom}(V,W), \quad A \mapsto \varphi_\gamma^{-1} L_A \varphi_\beta .$$