HW 2

1.) Let V be a vector space over a field F. Let $W < V$ be a subspace. Let $x, x', y, y' \in V$, $a \in F$, such that

$$x + W = x' + W \quad \text{and} \quad y + W = y' + W.$$

Show that

(a) $(x+y) + W = (x'+y') + W$ and

(b) $(ax) + W = (ax') + W$.

Hence our definition of addition and scalar multiplication on $V/W = \{x + W \mid x \in V\}$ is well defined.

2.) In the notation of 1., show that V/W is a vector space. (Check axioms (VS1)–(VS8).)

3.) Let $V = \mathbb{R}^3$ and $W = \operatorname{span}\left(\left\{\begin{pmatrix}1\\1\\1\end{pmatrix}\right\}\right)$, $U = \operatorname{span}\left(\left\{\begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\-1\\1\end{pmatrix}\right\}\right)$.

Give a geometric description of W, U, V/W and V/U.

4.) Let F be a field and $V = F^n$. Find a basis of

$$W = \left\{ \begin{pmatrix}x_1\\ \vdots \\ x_n\end{pmatrix} \in F^n \;\middle|\; \sum_{i=1}^{n} x_i = 0 \right\}.$$

5.) Let $F = \mathbb{F}_2$, X be a set and $V = \mathcal{P}(X) = \{S \subset X\}$. Show that $W = \{S \subset X \mid |S| \text{ is even}\}$.

(a) Show that W is a subspace of V.

(b) Compute $|V/W|$.

6.| Let V be a v.s. over a field F. Let $\{v_1, \dots, v_n\} \subset V$. Let

$$U = \text{span}(\{v_1, \dots, v_n\}) = \left\{ \sum_{i=1}^{n} a_i v_i \;\middle|\; a_i \in F \right\}$$

Show:

(a) $U \subset V$ is a subspace.

(b) Let $W \subset V$ be a subspace, such that $\{v_1, \dots, v_n\} \subset W$. Then $U \subset W$.

7.| Let F be a field. Find a basis of

(a) $V = \left\{ \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \;\middle|\; \sum_{i=1}^{n} a_i = 0 \right\} \subset F^n$

(b) $V = \left\{ A \in F^{n \times n} \;\middle|\; tr(A) = \sum_{i=1}^{n} A_{i,i} = 0 \right\}$

8.| Let X be a finite set. Find a basis of $\mathcal{P}(X)$ and $\mathcal{P}_{ev}(X)$

9.| Let F be a field. Show that the following are bases.

(a) $\left\{ e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \dots, e_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \right\} \subset F^n$

(b) $\left\{ E^{k,l} = (\delta_{i,k} \delta_{j,l})_{i,j} \;\middle|\; \begin{matrix} 1 \le i \le n \\ 1 \le j \le m \end{matrix} \right\} \subset F^{m \times n}$

(c) $\{1, x, x^2, \dots\} \subset F[X] = \{a_0 + a_1 x + \dots + a_n x_n \mid a_i \in F, n \in \mathbb{Z}_{\ge 0}\}$

10. Let V be a finite dimensional vector space over a field F. Let $n = \dim_F V$. Show

(a) If $\mathcal{G} \subset V$ generates V, then $|\mathcal{G}| \geq n$.

(b) If $L \subset V$ is linearly independent, then $|L| \leq n$

(c) If $L \subset V$ is linearly independent, then there is a basis $\beta \subset V$ of V, such that $L \subset \beta$. (L can be extended to a basis).