Practice Midterm 2

UCLA: Math 31A, Fall 2017

Instructor: Jens Eberhardt
Date: 08 October 2017

- This exam has 4 questions, for a total of 18 points.
- Please print your working and answers neatly.
- Write your solutions in the space provided showing working.
- Indicate your final answer clearly.
- You may write on the reverse of a page or on the blank pages found at the back of the booklet however these will not be graded unless very clearly indicated.
- Non programmable and non graphing calculators are allowed.

Name: ______________________________

ID number: ______________________________

Discussion section (please circle):

Day/TA	Allen Boozer	Ben Szczesny	Fan Yang
Tuesday	1A	1C	1E
Thursday	1B	1D	1F

Question	Points	Score
1	4	
2	4	
3	6	
4	4	
Total:	18	

1. (a) (2 points) Compute the derivative of the following functions.

   1. $f(x) = \sin\left(\sqrt[3]{\cos(x+1)} - x^3\right)$
   2. $f(x) = \tan\left(\frac{5x^2+11}{\cos(x)}\right)$

   (b) (2 points) Determine $\frac{dy}{dx}$ for points on the curve:

$$\sin(xy) + \cos(xy) = 1$$

Solution:

(a) 1. Let $u(x) = \sqrt[3]{\cos(x+1)} - x^3$. Using the chain rule we get

$$u'(x) = -1/3\cos(x+1)^{-2/3}\sin(x+1) - 3x^2.$$

Using the chain rule again, we obtain

$$\begin{aligned} f'(x) &= \cos(u(x))u'(x) \\ &= \cos\left(\sqrt[3]{\cos(x+1)} - x^3\right)\left(-1/3\cos(x+1)^{-2/3}\sin(x+1) - 3x^2\right) \end{aligned}$$

2. Let $u(x) = \frac{5x^2+11}{\cos(x)}$. Using the quotient rule we get

$$u'(x) = \frac{10x\cos(x) + (5x^2+11)\sin(x)}{\cos^2(x)}.$$

Using the chain rule, we obtain

$$\begin{aligned} f'(x) &= \sec^2(u(x))u'(x) \\ &= \sec^2\left(\frac{5x^2+11}{\cos(x)}\right)\frac{10x\cos(x) + (5x^2+11)\sin(x)}{\cos^2(x)} \end{aligned}$$

(b) We apply $\frac{d}{dx}$ to the equation and solve for $\frac{dy}{dx}$.

$$\begin{aligned} \frac{d}{dx}\left(\sin(xy) + \cos(xy)\right) &= \frac{d1}{dx} \\ \cos(xy)(x\frac{dy}{dx} + y) - \sin(xy)(x\frac{dy}{dx} + y) &= 0 \\ (\cos(xy) - sin(xy))(x\frac{dy}{dx} + y) &= 0 \\ x\frac{dy}{dx} + y &= 0 \\ \frac{dy}{dx} &= \frac{-y}{x} \end{aligned}$$

2. A bird flies in a straight line with a speed of $4m/s$ at a constant height of $30m$. At the moment $t = 0$ the bird is directly over your head.
   (a) (2 points) How fast is the distance between you and the bird changing at $t = 10s$?
   (b) (2 points) You have a rifle and keep it pointed at the bird. Determine the rate of change of the angle between your rifle and the ground at $t = 0$.

Solution:

(a) Denote the position of the bird on the straight line by $x(t)$ (with $x(0) = 0$), its height by h and its distance from you by $d(t)$. Hence

$$x = 4t$$
$$h = 30$$

and by Pythagoras's theorem we have

$$d(t) = \sqrt{h^2 + x^2} = \sqrt{30^2 + (4t)^2}$$

Hence

$$d'(t) = \frac{32t}{2\sqrt{30^2 + (4t)^2}}$$

and for $t = 10s$

$$d'(t) = \frac{320}{2\sqrt{30^2 + 40^2}} = \frac{320}{100} = 3.2m/s.$$

Hence the distance is changing by $3.2m/s$ at $t = 10s$.

(b) Denote said angle by $\theta(t)$. Then

$$x = h\tan(\pi/2 - \theta)$$

Applying $\frac{d}{dt}$ this yields

$$4 = \frac{dx}{dt} = \frac{d}{dt}h\tan(\pi/2 - \theta) = \frac{-h}{\cos^2(\pi/2 - \theta)}\frac{d\theta}{dt}$$

At $t = 0$ clearly $\theta = \pi/2$ and hence

$$\frac{d\theta}{dt} = -4\cos^2(\pi/2 - \theta)/h = -4/30$$

Hence θ changes by $-4/30rad/s$ at $t = 0$.

3. Consider the function

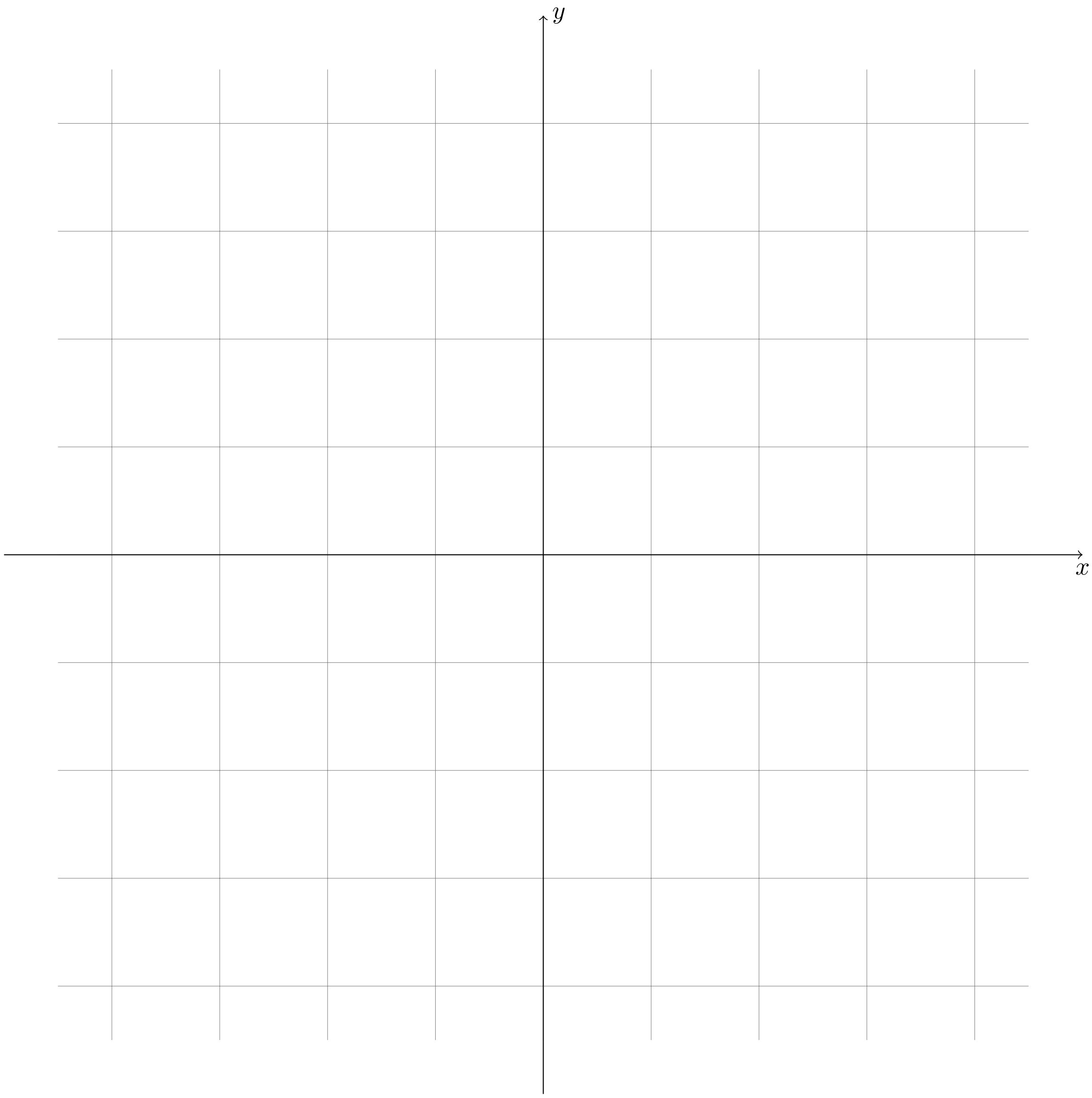
$$f(x) = \frac{x^4}{4} - 2x^2$$

   (a) (2 points) Determine the signs of f' and f''.

   (b) (1 point) Determine the local extrema and points of inflections of f.

   (c) (1 point) Determine the asymptotic behavior of f.

   (d) (2 points) Sketch the graph of f using the provided grid. Plot the transition points and connect them with the arcs corresponding to the sign combination of f' and f''.



Solution:

(a) We have

$$f'(x) = x^3 - 4x = x(x-2)(x+2)$$
$$f''(x) = 3x^2 - 4 = 3\left(x - \frac{2}{\sqrt{3}}\right)\left(x + \frac{2}{\sqrt{3}}\right)$$

and

Interval	Test value	sign of f'
$(-\infty, -2)$	$f'(-3) = -15$	$-$
$(-2, 0)$	$f'(-1) = 3$	$+$
$(0, 2)$	$f'(1) = -3$	$-$
$(2, \infty)$	$f'(3) = 15$	$+$

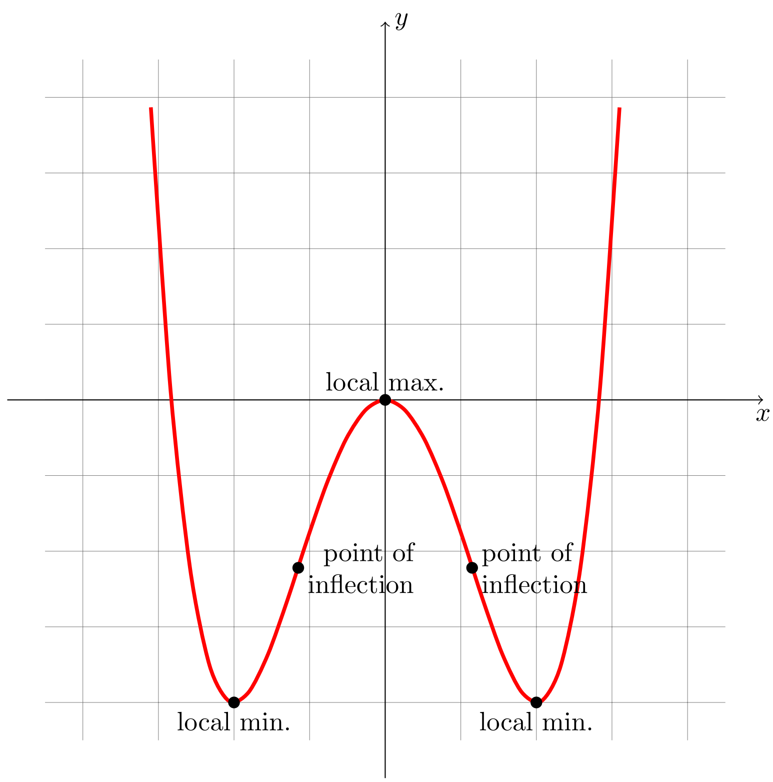
Interval	Test value	sign of f''
$(-\infty, -2/\sqrt{3})$	$f''(-2) = 8$	$+$
$(-2/\sqrt{3}, 2/\sqrt{3})$	$f''(0) = -4$	$-$
$(2/\sqrt{3}, \infty)$	$f''(2) = 8$	$+$

(b) The function f has local minima $f(-2) = f(2) = -4$, a local maximum $f(0) = 0$ and inflection points $f(-2/\sqrt{3}) = f(2/\sqrt{3}) = -20/9$.

(c) The function f has leading term $1/4x^4$ hence

$$\lim_{x\to\pm\infty} = \infty.$$

(d) Putting everything together we can sketch the graph of f:



4. (4 points) Find two positive real numbers x and y such that $x + y = 3$ and xy^2 is as big as possible.

Solution: We use the equation $x + y = 3$ to turn xy^2 into a function of just one variable.

$$\begin{aligned} x + y &= 3 \text{ is equivalent to} \\ y &= 3 - x. \end{aligned}$$

We substitute this into xy^2.

$$f(x) = xy^2 = x(3 - x)^2 = x^3 - 6x^2 + 9x$$

Our goal is to minimize $f(x)$ for $x \in [0, 3]$. For this we determine the critical points of $f(x)$.

$$f'(x) = 3x^2 - 12x^2 + 9 = 3(x^2 - 4x + 3) = 3(x - 1)(x - 3)$$

We hence have critical points at $c = 1$ and $c = 3$. To determine the maximum, we evaluate f at the critical points and the end points of the interval $[0, 3]$.

point	value
0	$f(0) = 0$
1	$f(1) = 4$
3	$f(3) = 0$

Hence $f(1) = 4$ is the maximum value which is obtained at $x = 1$ and $y = 2$.

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