Practice Midterm 2

UCLA: Math 115A, Fall 2017

Instructor: Jens Eberhardt
Date: 08 October 2017

- This exam has 4 questions, for a total of 23 points.
- Please print your working and answers neatly.
- Write your solutions in the space provided showing working.
- Indicate your final answer clearly.
- You may write on the reverse of a page or on the blank pages found at the back of the booklet however these will not be graded unless very clearly indicated.
- Non programmable and non graphing calculators are allowed.

Name: ______________________________

ID number: ______________________________

Question	Points	Score
1	7	
2	5	
3	5	
4	6	
Total:	23	

1. Let $T : V \to W$ be a **one-to-one** linear transformation between two finite dimensional vector spaces V and W over a field F.

   (a) (2 points) Let $v_1, v_2, \ldots, v_r \in V$ be linearly independent. Prove that $T(v_1), T(v_2), \ldots, T(v_r)$ are linearly independent as well.

   (b) (2 points) Prove that $\dim(V) \leq \dim(W)$. *Hint:* Use the rank-nullity formula.

   (c) (2 points) Assume that $\dim(V) = \dim(W)$. Show that T is onto. *Hint:* Use the rank-nullity formula again.

   (d) (1 point) Give an example of a linear map, which is one-to-one but not onto.

2. Let $\beta = \{1, x, x^2\}$ and $\beta' = \{1+x+x^2, x+x^2, x^2\}$ be bases of $V = P_2(\mathbb{R})$, the vector space of polynomials of degree less than or equal two.

   (a) (1 point) Determine the change of coordinate matrix from β' to β

   $$[I_V]_{\beta'}^{\beta}$$

   (b) (3 points) Determine the change of coordinate matrix from β to β'

   $$[I_V]_{\beta}^{\beta'}$$

   (c) (1 point) Let $T : V \to V$ be a linear transformation. Assume that you know $[T]_{\beta}^{\beta}$. What is $[T]_{\beta'}^{\beta'}$?

3. Let $T, U : V \to W$ be linear transformations between two finite dimensional vector spaces over a field F with bases β and γ, respectively.

   (a) (2 points) Prove that
   $$T + U : V \to W,\ (T + U)(v) = T(v) + U(v)$$
   is a linear transformation.

   (b) (1 point) Let $a \in F$. Assume that you know $A = [T]_\beta^\gamma$ and $B = [U]_\beta^\gamma$. Express is $[aT + U]_\beta^\gamma$ in terms of A and B.

   (c) (2 points) Let $v \in V$. Prove that
   $$Z = \{T : V \to W \,|\, T \text{ linear and } T(v) = 0\}$$
   is a subspace of
   $$\mathcal{L}(V, W) = \{T : V \to W \,|\, T \text{ linear}\}.$$

4. Let $V = P_3(\mathbb{R})$ and $W = M_{2,2}(\mathbb{R})$. Let

$$\beta = \{1, x, x^2, x^3\}$$
$$\gamma = \left\{ w_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, w_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, w_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, w_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

be the standard ordered bases. Consider the linear map $T : V \to$ defined by

$$T(ax^3 + bx^2 + cx^1 + d) = \begin{bmatrix} a+b & c+d \\ a+c & b+c \end{bmatrix}$$

(a) (2 points) Determine $A = [T]_\beta^\gamma$.

(b) (3 points) Prove that T is an isomorphism.

(c) (1 point) Prove that V and W are isomorphic without using T.

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