# Practice Final Exam

## UCLA: Math 115A

*Instructor:* Jens Eberhardt

- This exam has 6 questions, for a total of 60 points.
- Please print your working and answers neatly.
- Write your solutions in the space provided showing working.
- Indicate your final answer clearly.
- You may write on the reverse of a page or on the blank pages found at the back of the booklet however these will not be graded unless very clearly indicated.
- Non programmable and non graphing calculators are allowed.

Name: ______________________________

ID number: ______________________________

| Question | Points | Score |
|---|---|---|
| 1 | 10 | |
| 2 | 10 | |
| 3 | 10 | |
| 4 | 10 | |
| 5 | 10 | |
| 6 | 10 | |
| Total: | 60 | |

1. Consider the vector space $V = P_2(\mathbb{R})$ with its standard ordered basis

$$\beta = \{1, x, x^2\}$$

and the linear maps

$$T : P_2(\mathbb{R}) \to P_2(\mathbb{R}),\ T(f) = f(1) + f(-1)x + f(0)x^2$$
$$S : P_2(\mathbb{R}) \to P_2(\mathbb{R}),\ S(ax^2 + bx + c) = cx^2 + bx + a.$$

(a) (3 points) What is $[T]_\beta$ and $[S]_\beta$? Show that

$$[TS]_\beta = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

(b) (6 points) Compute $[(TS)^{-1}]_\beta$.

(c) (1 point) What is $(TS)^{-1}(x^2 + x + 1)$?

**Solution:**

(a) We have

$$T(1) = 1 + x + x^2$$
$$T(x) = 1 - x$$
$$T(x^2) = 1 + x$$

Hence

$$[T]_\beta = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Also clearly

$$[S]_\beta = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Using $[TS]_\beta = [T]_\beta [S]_\beta$ we get

$$[TS]_\beta = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

(b) We have $[(T+S)^{-1}]_\beta = [T+S]_\beta^{-1}$. We hence have to invert $[T+S]_\beta$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

Substract 2nd from 1st

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -2 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

Norm 2nd

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1/2 & -1/2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

Substract 2nd from 1st

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1/2 & 1/2 & 0 \\ 0 & 1 & 0 & 1/2 & -1/2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

Substract 3nd from 1st

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & 1/2 & -1 \\ 0 & 1 & 0 & 1/2 & -1/2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

Hence

$$[(TS)^{-1}]_\beta = \begin{bmatrix} 1/2 & 1/2 & -1 \\ 1/2 & -1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(c) We know

$$\begin{aligned} [(TS)^{-1}(x^2+x+1)]_\beta &= [(TS)^{-1}]_\beta [(x^2+x+1)]_\beta \\ &= \begin{bmatrix} 1/2 & 1/2 & -1 \\ 1/2 & -1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{aligned}$$

These are the coordinates of $x^2 = (TS)^{-1}(x^2+x+1)$.

2. Consider the matrix

$$A = \begin{bmatrix} 0 & 1 & -2 \\ 1 & 0 & -2 \\ 0 & 0 & -1 \end{bmatrix} \in M_{3,3}(\mathbb{R}).$$

(a) (2 points) Compute the characteristic polynomial of $A$ and determine the eigenvalues and their algebraic multiplicity.

(b) (6 points) Is $A$ is diagonalizable? If yes, compute a basis $\beta$ of eigenvectors of $A$.

(c) (2 points) Compute $[\mathrm{L}_A]_\beta$, where the $\mathrm{L}_A$ is the linear transformation given by

$$\mathrm{L}_A : \mathbb{R}^3 \to \mathbb{R}^3,\ v \mapsto Av.$$

**Solution:**

(a) We compute

$$\det(A - tI_3) = \det \begin{bmatrix} -t & 1 & -2 \\ 1 & -t & -2 \\ 0 & 0 & -1-t \end{bmatrix} = (t^2-1)(-1-t) = -(t+1)^2(t-1)$$

(b) The eigenvalues are the roots of the characteristic polynomial and hence $\lambda = 1, -1$ with multiplicity 1 and 2 respectively.

(c) We compute

$$\mathrm{N}(A - 1I_3) = \mathrm{Span}((1,1,0)^t)$$
$$\mathrm{N}(A - (-1)I_3) = \mathrm{Span}((-1,1,0)^t, (2,0,1)^t)$$

with our favorite algorithm (Wolfram Alpha). Hence

$$\beta = \{(1,1,0)^t, (-1,1,0)^t, (2,0,1)^t\}$$

is a basis of eigenvectors.

(d) A diagonal matrix with entries $1, -1, -1$ in an appropriate order.

3. Consider the vector space $V = \mathbb{R}^4$ with its standard inner product. Consider the linearly independent subset

$$S = \{w_1 = (1,0,1,0), w_2 = (1,1,1,1), w_3 = (2,2,0,2)\}.$$

(a) (6 points) Apply the Gram-Schmidt orthogonalization algorithm to $S$ to compute an orthogonal basis $\beta'$ of $\mathrm{Span}(S)$.

(b) (2 points) Use your result to compute an orthonormal basis $\beta$ of $\mathrm{Span}(S)$.

(c) (2 points) Let $x = (1,2,3,2) \in \mathrm{Span}(S)$. Compute the coordinate vector $[x]_\beta$.

**Solution:**

(a)

$$\begin{aligned} v_1 &= w_1 \\ &= (1,0,1,0) \\ v_2 &= w_2 - \frac{\langle w_2, v_1\rangle}{\langle v_1, v_1\rangle} v_1 = (1,1,1,1) - \frac{2}{2}(1,0,1,0) \\ &= (0,1,0,1) \\ v_3 &= w_3 - \frac{\langle w_3, v_1\rangle}{\langle v_1, v_1\rangle} v_1 - \frac{\langle w_3, v_2\rangle}{\langle v_2, v_2\rangle} v_2 = (2,2,0,2) - \frac{2}{2}(1,0,1,0) - \frac{4}{2}(0,1,0,1) \\ &= (1,0,-1,0) \end{aligned}$$

(b)

$$\begin{aligned} u_1 &= \frac{1}{||v_1||} v_1 = \frac{1}{\sqrt{2}}(1,0,1,0) \\ u_2 &= \frac{1}{||v_2||} v_2 = \frac{1}{\sqrt{2}}(0,1,0,1) \\ u_3 &= \frac{1}{||v_3||} v_3 = \frac{1}{\sqrt{2}}(1,0,-1,0) \end{aligned}$$

(c)

$$\begin{aligned} \langle x, v_1\rangle &= \frac{4}{\sqrt{2}} \\ \langle x, v_2\rangle &= \frac{4}{\sqrt{2}} \\ \langle x, v_3\rangle &= \frac{-2}{\sqrt{2}} \end{aligned}$$

Hence $[x]_\beta = \frac{1}{\sqrt{2}}(4,4,-2)$.

4. Let $S : U \to V$ and $T : V \to W$ be linear transformations between finite dimensional vector spaces $U, V$ and $W$ over a field $F$.
   (a) (2 points) Let $v_1, v_2, \ldots, v_n \in V$ be linearly independent and $\lambda_1, \lambda_2, \ldots, \lambda_n \in F$, such that $\lambda_i \neq 0$ for all $1 \leq i \leq n$. Show that also $\lambda_1 v_1, \lambda_2 v_2, \ldots, \lambda_n v_n$ are linearly independent.
   (b) (4 points) Let $v, w \in V$. Show that $\mathrm{span}(v, w) = \mathrm{span}(v, v + w)$.
   *Hint:* Proceed in two steps: Show that for all $x \in V$,
      1. $x \in \mathrm{span}(v, w)$ implies $x \in \mathrm{span}(v, v + w)$ and
      2. $x \in \mathrm{span}(v, v + w)$ implies $x \in \mathrm{span}(v, w)$.
   (c) (4 points) Assume that $\mathrm{R}(S) = \mathrm{N}(T)$, i.e. the range of $S$ is equal to the nullspace of $T$. Assume furthermore that $S$ is one-to-one and $T$ is onto. Show that

$$\dim V = \dim U + \dim W.$$

   *Hint:* Use the rank-nullity formula.

**Solution:**

(a) Let $a_1, a_2, \ldots, a_n \in F$ such that

$$a_1\lambda_1 v_1 + a_2\lambda_2 v_2 + \cdots + a_n\lambda_n v_n = 0.$$

Since $v_1, v_2, \ldots, v_n$ are linearly independent we get

$$a_1\lambda_1 = a_2\lambda_2 = \cdots = a_n\lambda_n = 0.$$

Since the $\lambda_i$ are nonzero this implies

$$a_1 = a_2 = \cdots = a_n = 0.$$

Hence $\lambda_1 v_1, \lambda_2 v_2, \ldots, \lambda_n v_n$ are linearly independent.

(b) 1.) Let $x \in \mathrm{span}(v, w)$. Hence there are $a, b \in F$ such that

$$x = av + bw = (a - 1)v + b(v + w).$$

Hence $x \in \mathrm{span}(v, v + w)$.
2.) Now let $x \in \mathrm{span}(v, v + w)$. Hence there are $a, b \in F$ such that

$$x = av + b(v + w) = (a + 1)v + bw.$$

Hence $x \in \mathrm{span}(v, w)$.
Putting 1.) and 2.) together we get $\mathrm{span}(v, w) = \mathrm{span}(v, v + w)$.

(c) Since $S$ is one-to-on we have

$$\dim(U) = \mathrm{rank}(S).$$

Since $T$ is onto we have

$$\dim(V) = \mathrm{nullity}(T) + \dim(W).$$

Since $\mathrm{R}(S) = \mathrm{N}(T)$ we have

$$\mathrm{nullity}(T) = \mathrm{rank}(S) = \dim(U).$$

Hence we get

$$\dim(V) = \dim(U) + \dim(W).$$

5. Let $V$ be a finite dimensional vector space over a field $F$. Recall that

$$\mathcal{L}(V, V) = \{T : V \to V \,|\, T \text{ is a linear transformation}\}$$

denotes the vector space of linear transformations from $V$ to $V$ (also called linear operators on $V$). Fix a vector $v \in V$ and define

$$Z = \{T \in \mathcal{L}(V, V) \,|\, T(v) = 0\}\,.$$

One calls $Z$ the *annihilator* of $v$ in $\mathcal{L}(V, V)$.

(a) (4 points) Show that $Z$ is a subspace of $\mathcal{L}(V, V)$.

(b) (2 points) Let $\lambda \in F$ such that $\lambda \neq 0$. Prove or disprove (by finding a counterexample) that

$$Z' = \{T \in \mathcal{L}(V, V) \,|\, T(v) = \lambda v\}$$

is a subspace of $\mathcal{L}(V, V)$.

(c) (2 points) Assume that $\beta = \{v_1, \ldots, v_n\}$ is an ordered basis of $V$, such that $v_1 = v$. Let $T \in \mathcal{L}(V, V)$. Show that $T \in Z$ if and only if the first column of $A = [T]_\beta$ equals 0.

(d) (2 points) Assuming $v \neq 0$, what is $\dim(Z)$?

**Solution:**

(a) One easily checks

$$\phi_v : \mathcal{L}(V, V) \to F, T \mapsto T(v)$$

is linear. Hence $Z = \mathrm{N}(\phi_v)$ is a subspace.

(b) Let $V = \mathbb{R}$, $v = 1$, $T_0 : \mathbb{R} \to \mathbb{R}, x \mapsto 0$ be the zero map and $\lambda = 1$. Then $T_0 \notin Z'$, since $T_0(v) = 0 \neq 1 = \lambda v$. Hence $Z'$ is not a subspace.

(c) We know that the first column in $A$ is given by $[T(v)]_\beta$. Assume that $T \in Z$, then $T(v) = 0$ and clearly $[T(v)]\beta = 0$. Assume that the first column of $A$ equals zero, i.e $[T(v)]_\beta = 0$. Then clearly $T(v) = 0$.

(d) Denote by $X \subset M_{n,n}(F)$ the subspace of all matrices whose first column is zero. Then clearly $\dim Z_m = n^2 - n$. By the last part

$$[-]_\beta : Z \to X, T \mapsto [T]_\beta$$

is an isomorphism (since $[-]_\beta : \mathcal{L}(V, V) \to M_{n,n}(F)$ is). Hence $\dim Z = \dim X = n^2 - n$.

6. Let $V$ be a finite dimensional vector space over $\mathbb{R}$ with an inner product $\langle x, y\rangle \in \mathbb{R}$ for $x, y \in V$.
   (a) (3 points) Let $\lambda \in \mathbb{R}$ with $\lambda > 0$. Show that

$$\langle x, y\rangle' = \lambda\langle x, y\rangle, \text{ for } x, y \in V,$$

   defines an inner product on $V$.
   (b) (2 points) Let $T : V \to V$ be a linear operator, such that

$$\langle T(x), T(y)\rangle = \langle x, y\rangle, \text{ for all } x, y \in V.$$

   Show that $T$ is one-to-one.
   (c) (2 points) Recall that the *norm* of a vector $x \in V$ is defined by $||x|| = \sqrt{\langle x, x\rangle}$. Show that

$$\langle x, y\rangle = \frac{1}{2}(||x + y||^2 - ||x||^2 - ||y||^2), \text{ for all } x, y \in V.$$

   Hence, the inner product can be recovered from the norm.
   *Hint:* Rewrite $\langle x + y, x + y\rangle$ using the properties of inner products. Use that $\langle x, y\rangle \in \mathbb{R}$ is a real number by assumption.
   (d) (3 points) Let $\beta = \{v_1, \ldots, v_n\}$ be a basis of $V$. The *Gram matrix* $G \in \mathrm{M}_{n,n}(\mathbb{R})$ of the inner product $\langle -, -\rangle$ with respect to $\beta$ is defined by

$$G_{i,j} = \langle v_i, v_j\rangle.$$

   Show that $G$ is invertible.

**Solution:**

(a) Show the 4 properties.

(b) Use non-degeneracy.

(c) Follow hint.

(d) Let $x \in V$ with $[x]_\beta = (a_1, \ldots, a_n)^t$. Then

$$\begin{aligned} G[x]_\beta &= (\sum_{j=1}^{n} G_{1,j}a_j, \ldots, \sum_{j=1}^{n} G_{n,j}a_j)^t \\ &= (\sum_{j=1}^{n}\langle v_1, v_j\rangle a_j, \ldots, \sum_{j=1}^{n}\langle v_n, v_j\rangle a_j)^t \\ &= (\langle v_1, \sum_{j=1}^{n} a_j v_j\rangle, \ldots, \langle v_n, \sum_{j=1}^{n} a_j v_j\rangle)^t \\ &= (\langle v_1, x\rangle, \ldots, \langle v_n, x\rangle)^t \end{aligned}$$

Hence $G[x]_\beta = 0 \Leftrightarrow x \in V^\perp = \{0\} \Leftrightarrow x = 0$, and $G$ is invertible.

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