Practice Final Exam

UCLA: Math 115A

Instructor: Jens Eberhardt

- This exam has 6 questions, for a total of 60 points.
- Please print your working and answers neatly.
- Write your solutions in the space provided showing working.
- Indicate your final answer clearly.
- You may write on the reverse of a page or on the blank pages found at the back of the booklet however these will not be graded unless very clearly indicated.
- Non programmable and non graphing calculators are allowed.

Name: ______________________________

ID number: ______________________________

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
Total:	60	

1. Consider the vector space $V = P_2(\mathbb{R})$ with its standard ordered basis

$$\beta = \{1, x, x^2\}$$

and the linear maps

$$T : P_2(\mathbb{R}) \to P_2(\mathbb{R}),\ T(f) = f(1) + f(-1)x + f(0)x^2$$
$$S : P_2(\mathbb{R}) \to P_2(\mathbb{R}),\ S(ax^2 + bx + c) = cx^2 + bx + a.$$

(a) (3 points) What is $[T]_\beta$ and $[S]_\beta$? Show that

$$[TS]_\beta = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

(b) (6 points) Compute $[(TS)^{-1}]_\beta$.

(c) (1 point) What is $(TS)^{-1}(x^2 + x + 1)$?

2. Consider the matrix

$$A = \begin{bmatrix} 0 & 1 & -2 \\ 1 & 0 & -2 \\ 0 & 0 & -1 \end{bmatrix} \in M_{3,3}(\mathbb{R}).$$

(a) (2 points) Compute the characteristic polynomial of A and determine the eigenvalues and their algebraic multiplicity.

(b) (6 points) Is A is diagonalizable? If yes, compute a basis β of eigenvectors of A.

(c) (2 points) Compute $[\mathrm{L}_A]_\beta$, where the L_A is the linear transformation given by

$$\mathrm{L}_A : \mathbb{R}^3 \to \mathbb{R}^3,\ v \mapsto Av.$$

3. Consider the vector space $V = \mathbb{R}^4$ with its standard inner product. Consider the linearly independent subset

$$S = \{w_1 = (1,0,1,0), w_2 = (1,1,1,1), w_3 = (2,2,0,2)\}\,.$$

   (a) (6 points) Apply the Gram-Schmidt orthogonalization algorithm to S to compute an orthogonal basis β' of $\mathrm{Span}(S)$.

   (b) (2 points) Use your result to compute an orthonormal basis β of $\mathrm{Span}(S)$.

   (c) (2 points) Let $x = (1,2,3,2) \in \mathrm{Span}(S)$. Compute the coordinate vector $[x]_\beta$.

4. Let $S : U \to V$ and $T : V \to W$ be linear transformations between finite dimensional vector spaces U, V and W over a field F.

   (a) (2 points) Let $v_1, v_2, \ldots, v_n \in V$ be linearly independent and $\lambda_1, \lambda_2, \ldots, \lambda_n \in F$, such that $\lambda_i \neq 0$ for all $1 \leq i \leq n$. Show that also $\lambda_1 v_1, \lambda_2 v_2, \ldots, \lambda_n v_n$ are linearly independent.

   (b) (4 points) Let $v, w \in V$. Show that $\text{span}(v, w) = \text{span}(v, v + w)$.
   Hint: Proceed in two steps: Show that for all $x \in V$,

   1. $x \in \text{span}(v, w)$ implies $x \in \text{span}(v, v + w)$ and
   2. $x \in \text{span}(v, v + w)$ implies $x \in \text{span}(v, w)$.

   (c) (4 points) Assume that $\text{R}(S) = \text{N}(T)$, i.e. the range of S is equal to the nullspace of T. Assume furthermore that S is one-to-one and T is onto. Show that

   $$\dim V = \dim U + \dim W.$$

   Hint: Use the rank-nullity formula.

5. Let V be a finite dimensional vector space over a field F. Recall that

$$\mathcal{L}(V,V) = \{T : V \to V \,|\, T \text{ is a linear transformation}\}$$

denotes the vector space of linear transformations from V to V (also called linear operators on V). Fix a vector $v \in V$ and define

$$Z = \{T \in \mathcal{L}(V,V) \,|\, T(v) = 0\}\,.$$

One calls Z the *annihilator* of v in $\mathcal{L}(V,V)$.

(a) (4 points) Show that Z is a subspace of $\mathcal{L}(V,V)$.

(b) (2 points) Let $\lambda \in F$ such that $\lambda \neq 0$. Prove or disprove (by finding a counterexample) that

$$Z' = \{T \in \mathcal{L}(V,V) \,|\, T(v) = \lambda v\}$$

is a subspace of $\mathcal{L}(V,V)$.

(c) (2 points) Assume that $\beta = \{v_1, \ldots, v_n\}$ is an ordered basis of V, such that $v_1 = v$. Let $T \in \mathcal{L}(V,V)$. Show that $T \in Z$ if and only if the first column of $A = [T]_\beta$ equals 0.

(d) (2 points) Assuming $v \neq 0$, what is $\dim(Z)$?

6. Let V be a finite dimensional vector space over $\mathbb{R}$ with an inner product $\langle x, y\rangle \in \mathbb{R}$ for $x, y \in V$.
   (a) (3 points) Let $\lambda \in \mathbb{R}$ with $\lambda > 0$. Show that

   $$\langle x, y\rangle' = \lambda\langle x, y\rangle, \text{ for } x, y \in V,$$

   defines an inner product on V.

   (b) (2 points) Let $T : V \to V$ be a linear operator, such that

   $$\langle T(x), T(y)\rangle = \langle x, y\rangle, \text{ for all } x, y \in V.$$

   Show that T is one-to-one.

   (c) (2 points) Recall that the *norm* of a vector $x \in V$ is defined by $||x|| = \sqrt{\langle x, x\rangle}$. Show that

   $$\langle x, y\rangle = \frac{1}{2}(||x + y||^2 - ||x||^2 - ||y||^2), \text{ for all } x, y \in V.$$

   Hence, the inner product can be recovered from the norm.
   Hint: Rewrite $\langle x + y, x + y\rangle$ using the properties of inner products. Use that $\langle x, y\rangle \in \mathbb{R}$ is a real number by assumption.

   (d) (3 points) Let $\beta = \{v_1, \ldots, v_n\}$ be a basis of V. The *Gram matrix* $G \in \mathrm{M}_{n,n}(\mathbb{R})$ of the inner product $\langle -, -\rangle$ with respect to β is defined by

   $$G_{i,j} = \langle v_i, v_j\rangle.$$

   Show that G is invertible.

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