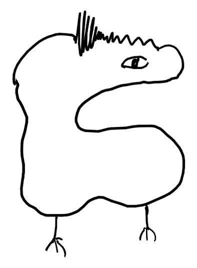
Notation: $f: X \to Y$ cont., $X, Y \in Top$

$f_{*,!} = Rf_{*,!}$, $f_{(*),(!)} = H^0 f_{*,!}$, $a: X \to pt$

$D(X) := Der^{(+)}(R\text{-Mod}/X)$ for $R \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \dots\}$

Recap: goal was: $H^i(X) \overset{\text{gen'ze}}{\leadsto} D(X)$, 6 functors

Example:



"the topological chicken"

$f^* \dashv f_*$
$\overset{L}{\otimes} \dashv RHom$ } 6 functors
$f_! \dashv f^!$ ← this we didn't do yet!

we did:
- Gysin triangle ⇝ H^*(CW complex)
- base change ⇝ projection formula ⇝ Künneth formula
- Grothendieck spectral sequence ⇝ Leray, Čech ⇝ MV l.e.s.
  ⇝ local-to-global: $\Gamma\mathcal{H}om = Hom$

still missing:
- Homology, pushforward functoriality (we did pullbacks on H^*)
- Poincaré duality
- Universal coefficients
- Alexander duality

All this comes from Verdier duality which is the existence of $f^!$ with $f_! \dashv f^!$.

Example: $X := \mathbb{R}$

$H^1 X = 0$ (algebraic duality; e.g. period iso) $H_1 X = 0$
$H^0 X = R$ $H_0 X = R$
(Poincaré duality) (Poincaré duality)
$H^1_! X = R$ (alg. duality) $H^!_1 X = R$
$H^0_! X = 0$ $H^!_0 X = 0$ } Borel-Moore homology

compact support

Def. $\omega_X := a^! R_{pt}$ dualising complex.
(remember $a^* R_{pt} = R_X$)

$H_\bullet(X) := H^{-\bullet} a_! a^! R_{pt}$ "sheaf homology"

$H^!_\bullet(X) := H^{-\bullet} a_* a^! R_{pt}$ "BM sheaf homology"

as before: $H^\bullet(X) := H^\bullet a_* a^* R_{pt}$, $H^\bullet_!(X) := H^\bullet a_! a^* R_{pt}$

Remark (intermediate definition):
if X nice enough, $\mathcal{C}^{-i}_{BM,X}(U) = \{$loc. $< \infty$ i-chains on $U\}$ is a complex of sheaves on X, which is soft, and $\mathcal{C}_{BM,X} \simeq \omega_X$.
We will prove something similar next time!

Plan:

§1. Co-Homology

§2. Milk the adjunction $f_! \dashv f^!$

§3. Prove it.

§2. Milking the adjunction

Def. $D_X := R\mathcal{H}om(\cdot, \omega_X)$ "Verdier duality"

Remark: $D_{pt} = RHom(\cdot, R)$ is usual duality of R-modules.
also, $D_X R_X = \omega_X$

Lemma: (1) $f_* R\mathcal{H}om(\cdot, f^! \cdot) = R\mathcal{H}om(f_! \cdot, \cdot)$ } follows from $f_! \dashv f^!$
"local Verdier duality"
(2) $f_* D_X = D_Y f_!$ } follows (1)
(3) $D^2_X R_X \cong R_X$ } only if X nice enough (already for $X = pt$, need finiteness condition on the modules!)

Corollary:
(1) $D_{pt}\, a_! a^* R = a_* D_X a^* R = a_* D_X R_X = a_* a^! R$
(2) $D_{pt}\, a_! a^! R = a_* D_X a^! R = a_* a^* R_X$ (X had to be nice enough)

for R a field, $H^i_!(X)^\vee \cong H^!_i(X)$ & $H^i(X) \cong H_i(X)^\vee$

for $R = \mathbb{Z}$, $0 \to Ext^1_{\mathbb{Z}}(H_{i-1}, \mathbb{Z}) \to H^i \to Hom(H_i, \mathbb{Z}) \to 0$
$0 \to Ext^1_{\mathbb{Z}}(H^{i-1}_!, \mathbb{Z}) \to H^!_i \to Hom(H^i_!, \mathbb{Z}) \to 0$ } s.e.s.

Lemma: Let X be an n-dim. manifold (over $\mathbb{R}$).
Then $\omega_X = or_X[n]$ (shifted orientation sheaf)

Proof 1: use Borel-Moore
Proof 2: we compare sections & use local Verdier duality:
for $U \cong \mathbb{R}^n$, $U \hookrightarrow X$:
$R\Gamma(U; a^! R) \cong RHom(R\Gamma_c(U; R_X), R)$
$\cong R[n]$
$a^! R = (U \mapsto Hom(H^n_c(U; R_X), R))^{shf}[n]$ as it turns out!

Corollary: if X is orientable, $\omega_X = R_X[n]$, so
$a_! a^! R = a_! R_X[n]$
$a_* a^! R = a_* R_X[n]$ } on cohomology get Poincaré duality: $H_i \cong H^{n-i}_!$ & $H^!_i \cong H^{n-i}$

§3 Proof of Verdier Duality

Basic strategy: reformulate the existence of a (right) adjoint (to $f_!$) to "representability":
$F: \mathcal{C} \to \mathcal{D}$ has a right adjoint iff
$Hom_{\mathcal{D}}(F-, D): \mathcal{C} \to Set$ is representable for all $D \in \mathcal{D}$

Keep in mind: $f_{(!)}$ is only left exact, not right exact, so it cannot have a right adjoint.

Q: When is a functor $G: Sh_X \to Ab^{op}$ rep'able?

Lemma (answer): iff G is cocontinuous.

Remark: of course Freyd's AFT works more generally!

Preparation: $\mathbb{Z}_{U \subset X} := j_! \mathbb{Z}_U$ for $j: U \hookrightarrow X$.
every $\mathcal{F} \in Sh_X$ is a colimit of such sheaves:
index cat $I :=$ ob: $\{(U,s) \mid U \subset X, s \in \mathcal{F}(s)\}$ Mor: restrictions
$T: I \to Sh_X$, $(U,s) \mapsto \mathbb{Z}_{U \subset X}$, morphisms via ext. by 0.
can show $\operatorname{colim} T \cong \mathcal{F}$ (or at least colim Something $\cong \mathcal{F}$)

Def.: for $\mathcal{K}$ flat & c-soft, $f^{\mathcal{K}}_{(!)}(\mathcal{F}) := f_{(!)}(\mathcal{F} \otimes \mathcal{K})$

Prop: $f^{\mathcal{K}}_!$ has a right adjoint $f^!_{\mathcal{K}}$

Lemma: $\mathcal{K}$ flat & c-soft $\Rightarrow \forall \mathcal{F}: \mathcal{F} \otimes \mathcal{K}$ is c-soft.

Pf of prop: $\cdot \otimes \mathcal{K}$ is exact, commutes with colimits, and maps c-soft to c-soft. As c-soft sheaves are adapted to $f_{(!)}$, $f^{\mathcal{K}}_{(!)}$ is exact. It also preserves direct sums $\Rightarrow f^{\mathcal{K}}_{(!)}$ cocontinuous $\Rightarrow \exists f^{(!)}_{\mathcal{K}}$. □

Theorem (Verdier duality) Let X, Y loc. cpt. Hausd. $\dim_{coho} X < \infty$ (so every sheaf has a finite injective resolution & there is a global bound on the length of a resolution.)
$f: X \to Y$, then $f_!$ has a right adjoint $f^!$

Proof: pick flat c-soft resolution
$0 \to \mathbb{Z}_X \to \mathcal{K}^1 \to \dots \to \mathcal{K}^d \to 0$
. . .

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part II §1 special cases & applications
§2 cosheaves

recap: $f: X \to Y$ cont. map of loc. compact Hausdorff spaces with $\dim_c X < \infty$,

VD: $f_!: D^+ X \rightleftarrows D^+ Y : f^!$

dualizing sheaf: for $a: X \to pt$, $\omega_X = a^! R_{pt} \in D^+ X$ (coefficients $R = \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \dots$)

If $R = k$ is a field, can choose $0 \to k_X \to \mathcal{K}^\bullet$ c-soft resolution and $\omega^i_X(U) = Hom(\Gamma_c(U, \mathcal{K}^{-i}), k)$ for $U \subset X$

1. Local cohom. & relative cohom.

we've seen: for $i: Z \hookrightarrow X$ closed embedding there's already underived:
$i_{(!)}: Ab_Z \rightleftarrows Ab_X : i^{(!)}$
where one has $i^{(!)} = i^{(*)} \Gamma_Z$ ← sections supported in Z

Lemma: $i^! = R i^{(!)} = i^{(*)} R\Gamma_Z$

Local coho: for $Z \hookrightarrow X$, $\Gamma_Z(X, \cdot)$ is left exact,
$H^i_Z(X, \mathcal{F}) = H^i R\Gamma_Z(X, \mathcal{F})$.

disting. Δ: $j: U = X \setminus Z \hookrightarrow X$, then
$i_* i^! \mathcal{F} \to \mathcal{F} \to j_* j^* \mathcal{F} \xrightarrow{[1]}$
$= i_!$ here $= j^!$ here
⇝ l.e.s. as usual.

Proposition: if X loc. contractible & paracompact, then
$H^i_Z(X; R_X) \cong H^i_{sing}(X, X \setminus Z; R)$
relative singular coho.

[there was a proof sketch on the blackboard which I didn't copy]

2. Examples for ω_X

Last time: X top. manifold of dim n, then ω_X is concentrated in one degree & $\mathcal{H}^{-n}(\omega_X) = or_X$ is a local system with stalk R. "orientation sheaf"
If X is an orientable C^0-manifold, and $R = \mathbb{Z}$, (usually C^0 also works)
$\{$isos $or_X \xrightarrow{\sim} \mathbb{Z}_X\} \xleftrightarrow{1:1} \{$orientations on $X\}$

Fact: $\omega_{X,x} \cong RHom(R\Gamma_{\{x\}}(X, R_X), R)$
for $R = k$ a field, using the proposition,
$H^i(\omega_x) \cong H^{-i}(X, X \setminus \{x\}; k)$.

Examples: a) $X = S^1 \vee S^1$ ∞
for $x \neq x_0$, $H^i(\omega_x) = \begin{cases} 0, & i \neq -1 \\ k, & i = -1 \end{cases}$
for $x = x_0$, $H^i(\omega_{x_0}) = \begin{cases} 0, & i \neq -1 \\ k^3, & i = -1 \end{cases}$
so ω_X is not a local system (as stalk ranks aren't loc. constant)

b) pinched torus $X =$
$x \neq x_0$: $H^i(\omega_x) = \begin{cases} 0, & i \neq -2 \\ k, & i = -2 \end{cases}$
$x = x_0$: $H^i(\omega_{x_0}) = \begin{cases} k, & i = -1 \\ k, & i = -2 \\ 0 & \text{else} \end{cases}$
so ω_X can't be a local system (as it's not concentrated in 1 degree)
⇝ "singularities are reflected in ω_X"

3. Relative case

rel. dualizing sheaf: $f: X \to Y$, then
$\omega_{X/Y} := f^! R_Y$

Lemma: ∃ natural $f^*(\cdot) \otimes \omega_{X/Y} \to f^!(\cdot)$

Def: $f: X \to Y$ "topological submersion" if f is locally a projection and $\forall x \in X\ \exists U \subset X$, $p \in U$, $\exists V \subset Y$, $f(p) \in V$ s.th. $\exists U \xrightarrow{\sim} V \times \mathbb{R}^\ell$ with
$U \xrightarrow{\sim} V \times \mathbb{R}^\ell$, $f \downarrow$, $\swarrow pr_1$, V & we call ℓ "fiber dim."

Example: submersions of smooth mf's are also topological submersions (at least if $\dim Y$ is smooth)

Lemma: $f: X \to Y$ top. subm. of fiber dim ℓ, then
1) $\mathcal{H}^k(\omega_{X/Y}) = 0$ $k \neq -\ell$
$\mathcal{H}^{-\ell}(\omega_{X/Y}) =: or_{X/Y}$ is a loc. sys. with stalk R. "relative orientation sheaf"
2) $f^*(\cdot) \otimes \omega_{X/Y} \to f^!(\cdot)$ is an iso.

And now, let's use cosheaves

this lovely series of 4 45-minute long talks was delivered by Jens & Florian in their wonderful seminar on sheaf cohomology.

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part II.2: cosheaves, Verdier duality, homology

Def. a functor $\mathcal{F}: Open(X) \to R\text{-Mod}$ with
(MV) $\bigoplus_{i,j} \mathcal{F}(U_i \cap U_j) \to \bigoplus_i \mathcal{F}(U_i) \to \mathcal{F}(\bigcup U_i) \to 0$ exact
for all $\{U_i\} \subset Open(X)$
is called a <u>cosheaf</u>.

Examples: 1) let X loc. path-connected,
$\underline{R}^X(U) := H^{sing}_0(U; R)$ "constant cosheaf"

2) cosheaf of singular limit chains:
$\mathfrak{S}(U) := \operatorname{colim}(S_i(U) \xrightarrow{B} S_i(U) \xrightarrow{B} \dots)$
↑ singular i-chains, B is barycentric subdivision

3) $\Gamma_! \mathcal{F}$ is a c-soft sheaf
$(\Gamma_! \mathcal{F})(U) = \Gamma_!(U; \mathcal{F})$
$0 \to \mathcal{F}_{U \cap V} \to \mathcal{F}_U \oplus \mathcal{F}_V \to \mathcal{F}_{U \cup V} \to 0$

?

Remark: cosheafification is non-trivial, as for Grothendieck's $\mathcal{F}^+$-construction, one needs certain (co)limits in $(R\text{-Mod})^{op}$ to commute.
In R-Mod, this is no longer true!

§1 "Poincaré duality"

Thm: X loc. compact, then
$\Gamma_!: \{$c-soft sheaves$\}/_\sim \xleftrightarrow{\sim} \{$flabby cosheaves$\}/_\sim : (-)^!$
where $\mathcal{F}$ cosheaf is flabby if $\forall U$: $\mathcal{F}(U) \hookrightarrow \mathcal{F}(X)$ injective.

Def.: $\mathcal{F}(X, U) := \mathcal{F}(X)/\mathcal{F}(U)$ for a flabby cosheaf

Lemma: flabby cosheaves have the excision property, i.e.
$\forall A \hookrightarrow U \hookrightarrow X$, $\mathcal{F}(X, U) \cong \mathcal{F}(X \setminus A, U \setminus A)$.

Def + Remark: $\mathcal{F}^!(U) = \varprojlim_{K \subset U} \mathcal{F}(U, U \setminus K)$
is a c-soft sheaf,
where $\mathcal{F}^!(V) \xrightarrow{res} \mathcal{F}^!(U)$ for $U \subset V$
is given by using the excision property.

Examples: $H^{BM}_i(X) = \varprojlim_{K \subset X} H_i(X, X \setminus K)$.

$H^i(X; \mathfrak{S}^!_\bullet) = H^i(\Gamma(X; \mathfrak{S}^!_\bullet)) = H^{BM}_i(X)$ if X is 2nd countable

$H^i_!(X; \mathfrak{S}^!_\bullet) = H^i(\Gamma_!(X; \mathfrak{S}^!_\bullet)) = H^{sing}_i(X)$

§2 Algebraic duality (let $R = k$ a field now)

Def + Remark: for $\mathcal{F}$ a cosheaf,
$A\mathcal{F}(U) := \mathcal{F}(U)^\vee := Hom_k(\mathcal{F}(U), k)$
defines a sheaf $A\mathcal{F}$. But: the dual A of a sheaf is not a cosheaf, since dual$(\oplus) = \Pi$, but dual$(\Pi) \neq \oplus$!

Note: $D_X \underline{R}_X = \omega_X = RHom(R\Gamma_!(-; \underline{R}_X), R)$
so Verdier duality = algebraic duality ∘ Poincaré duality.

Examples: 1) $A\underline{k}^X_X = \underline{k}_X$
2) $A\mathfrak{S}_i = C^i_{sing}$
3) $D_X \underline{k}_X = \omega_X = A\Gamma_! C^i_{sing} = A\Gamma_! A\mathfrak{S}_\bullet$

Now let for (3) X be loc. contractible.

§3 Verdier duality

as $(\,)^! \hat{=}$ limits, $\Gamma_! \hat{=}$ colimits, duality should exchange them.

Thm (Soergel): let X loc. compact Hausdorff & loc. polyhedron-like, then
$\mathfrak{S}^!_\bullet = A\Gamma_! A\mathfrak{S}_\bullet$ ($= \omega_X$ for X loc. contractible)

Corollary: $D^2_X \underline{k}_X = D_X A\Gamma_! A\mathfrak{S}_\bullet = A\Gamma_! A\Gamma_! A\mathfrak{S}_\bullet$
$\overset{\text{Thm}}{=} A\Gamma_! \mathfrak{S}^!_\bullet = A\mathfrak{S}_\bullet = \underline{k}_X$