Graded Parabolic Induction for Category $\mathcal{O}$

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Motivation for Category $\mathcal{O}$

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Let us turn this into **algebra**!

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$$G \curvearrowright \mathrm{Fun}(X, \mathbb{C})$$

- semi-simple Lie group,
  e.g. $\mathrm{SL}_n(\mathbb{R})$, $\mathrm{SO}(n, m)$, $\mathrm{SL}_n(\mathbb{C}) \ldots$
- *acting on some space of functions,*
  e.g. $\mathscr{C}^\infty(X), L^p(X), \ldots$

Let us turn this into **algebra**!

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- no canonical choice of Fun$(X, \mathbb{C})$,
- still in the realm of functional/harmonic analysis: full of $\int$ and $\sum^{\infty}$.

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Solution by *Harish-Chandra (1960's)*:

$$(\mathfrak{g}, K) \rightsquigarrow$$

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Solution by *Harish-Chandra (1960's)*:

$$(\mathfrak{g}, K) \looparrowright \mathrm{Fun}(X, \mathbb{C})_K$$

- $\mathfrak{g} = \mathrm{Lie}(G)$, the Lie algebra,
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- acting on K-finite vectors in $\mathrm{Fun}(X, \mathbb{C})$,
  i.e. $v \in \mathrm{Fun}(X, \mathbb{C})$ with $\dim_{\mathbb{C}} \langle Kv \rangle < \infty$

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This is called a *Harish-Chandra module*.

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Theorem (Casselman-Wallach, Kashiwara-Schmid):

You can functorially *globalize* a Harish-Chandra module to G.

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Special case: G complex Lie group, e.g. $G = \mathrm{SL}_n(\mathbb{C})$. Then

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Category $\mathcal{O}$ is **purely algebraic** and just depends on $\mathfrak{g}$.

Insight (Bernstein-Gelfand-Gelfand 1971):

You can understand $\mathscr{HC}(\mathfrak{g}, K)$ via just using $\mathcal{O}(\mathfrak{g})$.

Definition of Category $\mathcal{O}$

Cartan   Borel

$$\begin{tikzcd}
\mathfrak{h} \arrow[r, hook] \arrow[d, phantom, "\sqcap"] & \mathfrak{b} \arrow[r, hook] \arrow[d, phantom, "\Vert"] & \mathfrak{g} \arrow[d, phantom, "\sqcap"] \\
\begin{bmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{bmatrix} \arrow[r, hook] & \begin{bmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{bmatrix} \arrow[r, hook] & \mathfrak{sl}_n(\mathbb{C})
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$$\begin{array}{ccccc} \mathfrak{h} & \hookrightarrow & \mathfrak{b} & \hookrightarrow & \mathfrak{g} \\ \rotatebox{90}{$\subset$} & & \| & & \rotatebox{90}{$\subset$} \\ \begin{bmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{bmatrix} & \hookrightarrow & \begin{bmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{bmatrix} & \hookrightarrow & \mathfrak{sl}_n(\mathbb{C}) \end{array}$$

$$\{ L(\lambda) | \lambda \in \mathfrak{h}^* \}$$

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$$\mathcal{O}(\mathfrak{g}) \stackrel{\text{def}}{=} \langle L(\lambda) | \lambda \in \mathfrak{h}^* \rangle_{ext}.$$

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  1. $H.v^+ = \lambda(H)v^+$ for $H \in \mathfrak{h}$ and
  2. $\mathfrak{b}.v^+ \subseteq \langle v^+ \rangle_{\mathbb{C}}$.
- Take all extensions of those! (in the category of weight modules)

Parabolic Induction

Parabolic induction: inductively construct modules in $\mathcal{O}(\mathfrak{g})$:

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Levi Parabolic

$$
\begin{tikzcd}
\mathfrak{l} \arrow[r, hook] \arrow[d, equal] & \mathfrak{p} \arrow[r, hook] \arrow[d, equal] & \mathfrak{g} \arrow[d, equal] \\
\begin{bmatrix} \boxed{*} & & 0 \\ & \ddots & \\ 0 & & \boxed{*} \end{bmatrix} \arrow[r, hook] & \begin{bmatrix} \lfloor * & & * \\ & \ddots & \\ 0 & & \lfloor * \end{bmatrix} \arrow[r, hook] & \mathfrak{sl}_n(\mathbb{C})
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$$\begin{array}{c} \mathcal{O}(\mathfrak{l}) \\ \rotatebox{90}{$\in$} \\ M \end{array}$$

- Category $\mathcal{O}$ for $\mathfrak{l}$ (block matrices)

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$$\begin{tikzcd}
\mathcal{O}(\mathfrak{l}) \arrow[rr, "\mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}}"] & & \mathcal{O}(\mathfrak{g}) \\
M \arrow[u, phantom, "\cup"] \arrow[r, mapsto] & M \arrow[r, mapsto] & \mathbb{C}[\mathfrak{n}] \otimes_{\mathbb{C}} M \arrow[u, phantom, "\cup"]
\end{tikzcd}$$

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Goal: Understand parabolic induction better!

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Theorem (E.):

Parabolic induction $\mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}} : \mathcal{O}(\mathfrak{l}) \to \mathcal{O}(\mathfrak{g})$ can be understood in terms of a *finite* family of functors

$$\mathrm{Ind}_w : \mathcal{O}_0(\mathfrak{l}) \to \mathcal{O}_0(\mathfrak{g})$$

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subcategories generated by *finitely* many simples: *Principal blocks*

Geometric Representation Theory

Second step: apply methods from geometric representation theory.

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Solve problems in representation theory using geometric and cohomological methods!

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Solve problems in representation theory using geometric and cohomological methods!

Kazdhan-Lusztig (1979):

$$\mathcal{O}(\mathfrak{g}) \leftrightsquigarrow \text{Geometry of Schubert varieties.}$$

Flag variety

$G^\vee = \mathrm{SL}_n(\mathbb{C}) \supset B = \begin{bmatrix} \ddots & * \\ 0 & \ddots \end{bmatrix}$ Langlands dual group to $\mathfrak{g} \supset \mathfrak{b}$.

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Flag variety:

$$\begin{aligned} X^\vee = G^\vee / B &= \{0 \subseteq V_1 \subseteq \cdots \subseteq V_{n-1} \subseteq \mathbb{C}^n \,|\, \dim V_i = i\} \\ &= \biguplus_{w \in S_n} BwB/B \end{aligned}$$

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- For $w \in S_n$, $\overline{BwB/B}$ is called a Schubert variety.

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- For $w \in S_n$, $\overline{BwB/B}$ is called a Schubert variety.
- Singularities in Schubert varieties reflect complexity of simple modules in $\mathcal{O}(\mathfrak{g})$.

Graded Geometric Parabolic Induction

Soergel-Wendt 2015: Derived category of certain motives (upgraded sheaves) on $X^\vee$.

$$\mathrm{Der}^{\mathbb{Z},b}_{(B)}(X^\vee)$$

Graded Geometric Parabolic Induction

Soergel-Wendt 2015:
Derived category of certain motives (up-graded sheaves) on $X^\vee$.

Graded version of derived category $\mathcal{O}$.

$$\begin{array}{c} \mathrm{Der}^{\mathbb{Z},b}_{(B)}(X^\vee) \\ \downarrow v \\ \mathrm{Der}^b(\mathcal{O}_0(\mathfrak{g})) \end{array}$$

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Graded version of derived category $\mathcal{O}$.

$$\begin{tikzcd}
\mathrm{Der}^{\mathbb{Z},b}_{(B)}(X^\vee) \arrow[d, "v"] & \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}(M, N\langle n\rangle) \arrow[d, equal] \\
\mathrm{Der}^b(\mathcal{O}_0(\mathfrak{g})) & \mathrm{Hom}(v(M), v(N))
\end{tikzcd}$$

Graded Geometric Parabolic Induction

Same for $\mathfrak{l}$ and the associated flag variety $Y^\vee$.

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\downarrow v & & \downarrow v & & \| \\
\mathrm{Der}^b(\mathcal{O}_0(\mathfrak{l})) & \xrightarrow{\mathrm{Ind}_w} & \mathrm{Der}^b(\mathcal{O}_0(\mathfrak{g})) & & \mathrm{Hom}(v(M), v(N))
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Is there a geometric and graded version of parabolic induction?

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Is there a geometric and graded version of parabolic induction?

Yes! Use motivic six functor formalism!

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$$
\begin{tikzcd}
\mathrm{Der}^{\mathbb{Z},b}_{(C)}(Y^\vee) \arrow[r, "h_{w,*}\mathrm{pr}^!"] \arrow[d, "v"] & \mathrm{Der}^{\mathbb{Z},b}_{(B)}(X^\vee) \arrow[d, "v"] & \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}(M, N\langle n\rangle) \arrow[d, equal] \\
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\end{tikzcd}
$$

Theorem (E.):

There are maps $Y^\vee \xleftarrow{\mathrm{pr}} Y^\vee \times \mathbb{C}^{l(w)} \xrightarrow{h_w} X^\vee$ such that $? = h_{w,*}\mathrm{pr}^!$.

Preprint

Jens Niklas Eberhardt, *Graded parabolic induction and stratified mixed Tate motives*, arXiv:1603.00327 (2016)

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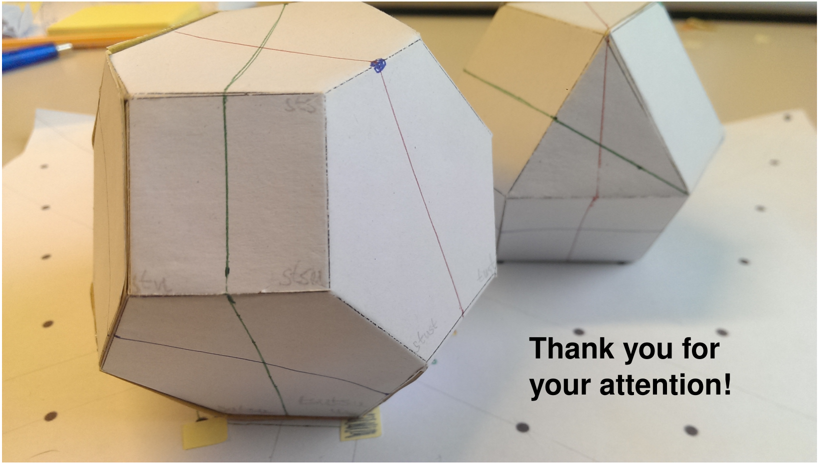
Jens Niklas Eberhardt and Shane Kelly, *A motivic six functor formalism for the graded modular category $\mathcal{O}$*, in preparation


Thank you for
your attention!