pt = Spec $\mathbb{F}_p$, coeff. = $\mathbb{Q}$

K-motives and Koszul duality

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split alg. torus $\boxed{T \overset{\text{duality}}{\longleftrightarrow} T^\vee}$ dual torus

character lattice $\boxed{X(T) = Y(T^\vee)}$ cocharacter lattice

T-equivariant alg. K-theory $\boxed{K_0^T(pt) = R = \mathbb{Q}[\pi_1(T^\vee(\mathbb{C}))]}$ group algebra of fundamental group

$R := \mathbb{Q}[X(T)]$

constant T-eq. K-motives $\boxed{DK^T_{const}(pt) = D^b(R\text{-mod}) = D^b(LC(T^\vee(\mathbb{C})))}$ der. cat. of locally const. sheaves

$\boxed{\begin{array}{ll} \text{Today:} & * \text{ K-motives??} \\ & * \; T \rightsquigarrow \mathfrak{g} \end{array}}$

II K-motives

(constructible sheaves : singular cohomology)

$=$

(K-motives : alg. K-theory)

variety $\quad X \longmapsto DK(X)$ $\quad$ triangulated category of K-motives on X

* 6 functor formalism $(f^*, f_*, f_!, f^!, \otimes, \mathcal{H}om)$
* equivariant version $DK^T(X)$, for $T \curvearrowright X$

for X regular $\quad \mathrm{Hom}_{DK^T(X)}(\mathbb{Q}, \mathbb{Q}[n]) = K_n^T(X)$ $\quad$ T-equivariant (higher) alg. K-theory

III "Intersection K-theory complexes"

$$\boxed{DK^T_{const}(X) \overset{\text{def.}}{=} \langle \mathbb{Q} \rangle_\Delta \subset DK^T(X)}$$

$$\boxed{DK^T_{const}(pt) = D^b(R\text{-mod})} \Leftarrow \boxed{\begin{aligned}&\mathrm{Hom}_{DK^T(pt)}(\mathbb{Q}, \mathbb{Q}[n]) \\ &= K^T_n(pt) = \begin{cases} R & n=0 \\ 0 & \text{else}\end{cases}\end{aligned}}$$

$$\mathbb{Q} \longleftrightarrow R$$

$$\boxed{DK^T_{\mathcal{S}}(X) \overset{\text{def}}{=} \{ M \mid M|_{X_s} \text{ constant} \}}$$

$$\boxed{\begin{aligned}&X = \biguplus_{s \in \mathcal{S}} X_s, \quad X_s \cong \mathbb{A}^n \\ &\text{affinely stratified} \\ &+ \text{assumptions}\end{aligned}} \Rightarrow$$

Thm (E.)

- $DK^T_{\mathcal{S}}(X)$ has "weight structure"
- $DK^T_{\mathcal{S}}(X)_{w=0}$ pure K-motives
- $DK^T_{\mathcal{S}}(X) \xrightarrow{\sim} K^b(DK^T_{\mathcal{S}}(X)_{w=0})$

analogous to

- intersection coh. complexes
- parity sheaves

"Intersection K-th. complexes"

IV Koszul duality à la Beilinson–Ginzburg–Soergel

$$\boxed{T \subset G \xleftrightarrow[\text{dual}]{\text{Langlands}} T^\vee \subset G^\vee}$$

mixed sheaves: * Tate twist (1)
* "graded version" of constr. sheaves

$D^b(\mathcal{O}_0^{\mathbb{Z}}(\mathfrak{g}))$ graded cat. $\mathcal{O}$ of $\mathrm{Lie}(G(\mathbb{C}))$

$- \otimes_{\mathbb{Q}} \mathbb{C}$

$$D^{mix}_{(B)}(G/B) \xleftrightarrow[\text{duality}]{\text{Koszul}} D^{mix}_{(B^\vee)}(G^\vee/B^\vee)$$

IC-complex $\longleftrightarrow$ perverse tilting sheaf

$(1)[2] \longleftrightarrow (1)$

constant along B-orbits (Bruhat cells)

Not true without **mix**!!

V Ungraded Koszul duality (E.)

Thm (E.)

$$\begin{tikzcd}
(1)[2] \arrow[d, mapsto] & D^{mix}_{(B)}(G/B) \arrow[r, leftrightarrow] \arrow[d, "real"] & D^{mix}_{(B^\vee)}(G^\vee/B^\vee) \arrow[d, "real"] & (1) \arrow[d, mapsto] \\
id & DK_{(B)}(G/B) \arrow[r, leftrightarrow] & D_{(B^\vee)}(G^\vee/B^\vee) & id
\end{tikzcd}$$

$(1)[2] \longleftrightarrow (1)$

$H^{i,j}(\mathbb{P}^1) = \begin{array}{c} \mathbb{Q} \\ \mathbb{Q} \end{array}$

$K_i(\mathbb{P}^1) = \mathbb{Q} \oplus \mathbb{Q}$

constructible sheaves on $(G^\vee/B^\vee)(\mathbb{C})^{an}$

VI Work in progress

Thm/Conj. [E.]

$$
\begin{tikzcd}
DK^T_{(B)}(G/B) \arrow[rr, leftrightarrow] \arrow[dr, "\sim"', "\text{weight complex functor}"] & & D_{T^\vee\text{-mon}}(G^\vee/U^\vee) \\
& K^b(R\text{-SBim}) \arrow[ur, "\sim", "\text{realisation functor}"'] &
\end{tikzcd}
$$

K-theory Soergel bimodules

etc. . . .

VI Work in progress

THANK YOU

Thm/Conj. (E.)

$$\begin{tikzcd}
DK^{T}_{(B)}(G/B) \arrow[rr, leftrightarrow] \arrow[dr, "\sim"', "\text{weight complex functor}"] & & D_{T^\vee\text{-mon}}(G^\vee/U^\vee) \\
& K^b(R\text{-}SBim) \arrow[ur, "\sim", "\text{realisation functor}"'] &
\end{tikzcd}$$

K-theory Soergel bimodules

etc ...

rk in progress

THANK YOU

$$\ldots_{(B)}(\mathcal{G}/B) \longleftrightarrow D_{T^\vee\text{-mon}}(\mathcal{G}^\vee/U^\vee)$$

realisation functor

weight complex functor

$$K^b(R\text{-}SBim)$$

K-theory Soergel bimodul

etc

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THANK YOU

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realisat…
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