

pt = Spec $\overline{\mathbb{F}}_p$, coeff. = \mathbb{Q}

K-motives and Koszul duality

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split alg. torus $\left[T \xleftrightarrow{\text{duality}} T^\vee \right]$ dual torus

character lattice $\left[X(T) = Y(T^\vee) \right]$ cocharacter lattice

T-equivariant alg. K-theory $\left[K_0^T(\text{pt}) = \mathbb{R} = \mathbb{Q}[\pi_1(T^\vee(\mathbb{C}))] \right]$ group algebra of fundamental group
 $\mathbb{R} := \mathbb{Q}[\langle \gamma \rangle]$

constant T-eg. K-motives $\left[DK_{\text{const}}^T(\text{pt}) = \mathcal{D}^b(\mathbb{R}\text{-mod}) = \mathcal{D}^b(\text{LC}(T^\vee(\mathbb{C}))) \right]$ der. cat. of locally const. sheaves

Today: * K-motives??
* $T \rightsquigarrow \mathfrak{g}$

II K-motives

(constructible sheaves : singular cohomology)

(K-motives : alg. K-theory)

variety

$$X \longmapsto \mathcal{D}K(X)$$

triangulated category
of K-motives on X

- * 6 functor formalism $(f^*, f_*, f_!, f^!, \otimes, \text{Hom})$
- * equivariant version $\mathcal{D}K^T(X)$, for $T \curvearrowright X$

for X
regular

$$\text{Hom}_{\mathcal{D}K^T(X)}(\mathcal{O}, \mathcal{O}[n]) = K_n^T(X)$$

T-equivariant
(higher) alg. K-theory

III "Intersection K-theory complexes"

$$\mathbb{D}K_{\text{const}}^T(X) \stackrel{\text{def.}}{=} \langle \mathcal{Q} \rangle_{\Delta} \subset \mathbb{D}K^T(X)$$

$$\mathbb{D}K_{\text{const}}^T(\text{pt}) = \mathbb{D}^b(\mathbb{R}\text{-mod})$$

$$\mathcal{Q} \leftrightarrow \mathbb{R}$$

$$\text{Hom}_{\mathbb{D}K^T(\text{pt})}(\mathcal{Q}, \mathcal{Q}(n))$$

$$= K_n^T(\text{pt}) = \begin{cases} \mathbb{R} & n=0 \\ 0 & \text{else} \end{cases}$$

$$\mathbb{D}K_g^T(X) \stackrel{\text{def.}}{=} \{M \mid M|_{X_s} \text{ constant}\}$$

$X = \bigoplus_{s \in S} X_s, X_s \cong \mathbb{A}^n$
 affinely stratified
 + assumptions

- Thm(E.)
- * $\mathbb{D}K_g^T(X)$ has "weight structure"
 - * $\mathbb{D}K_g^T(X)_{w=0}$ pure K-motives
 - * $\mathbb{D}K_g^T(X) \simeq K^b(\mathbb{D}K_g^T(X)_{w=0})$

analogous to

- * intersection coh. complexes
- * parity sheaves

"Intersection K-th. complexes"

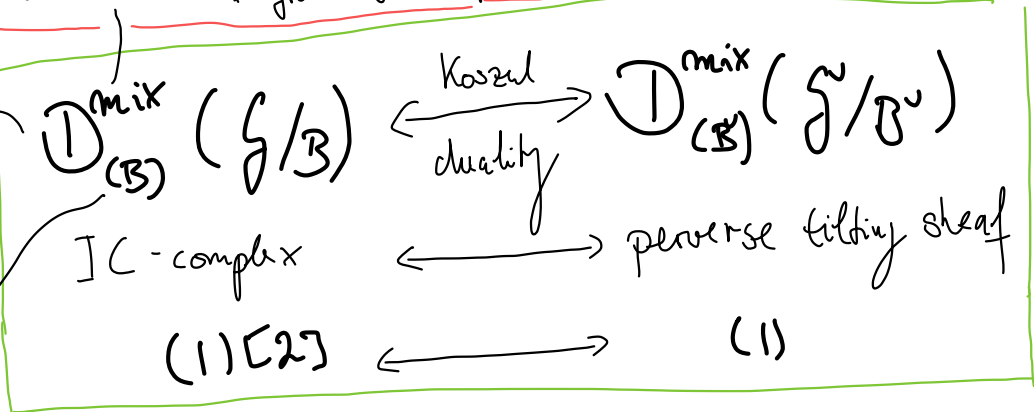
IV Koszul duality à la Beilinson-Ginzburg-Soergel

$$T \subset \mathfrak{g} \xleftrightarrow[\text{dual}]{\text{Langlands}} T^\vee \subset \mathfrak{g}^\vee$$

* Tate twist (1)
 mixed sheaves: * "graded version" of constr. sheaves

$D^b(\mathcal{O}_0^Z(\mathfrak{g}))$
 graded cat. \mathcal{O}
 of $\text{Lie}(\mathfrak{g}(C))$

constant along
 B -orbits
 (Bruhat cells)



Not true without mix!!

V Ungraded Koszul duality (E.)

Thm (E.)

$$(1) \Leftrightarrow (2) \quad \longleftrightarrow \quad (1)$$

$$(1) \Leftrightarrow (2) \quad \mathbb{D}_{(B)}^{\text{mix}}(\mathcal{F}/B) \quad \longleftrightarrow \quad \mathbb{D}_{(B^v)}^{\text{mix}}(\mathcal{F}^v/B^v) \quad (1)$$

real

real

id

$$\mathbb{D}K_{(B)}(\mathcal{F}/B) \quad \longleftrightarrow \quad \mathbb{D}_{(B^v)}(\mathcal{F}^v/B^v) \quad \text{id}$$

$$\begin{aligned} H^i(\mathcal{P}^v) &= \mathbb{Q} \\ K_*(\mathcal{P}^v) &= \mathbb{Q} \otimes \mathbb{Q} \end{aligned}$$

constructible sheaves
on $(\mathcal{F}^v/B^v)(\mathbb{A}^1)^{\text{an}}$

VI Work in progress

$$\text{Thm/Conj. [F.]} \quad \mathcal{DK}_{(\mathcal{B})}^T(g/\mathcal{B}) \longleftrightarrow \mathcal{D}_{T\text{-mon}}(g^v/u^v)$$

weight complex
functor

realisation
functor

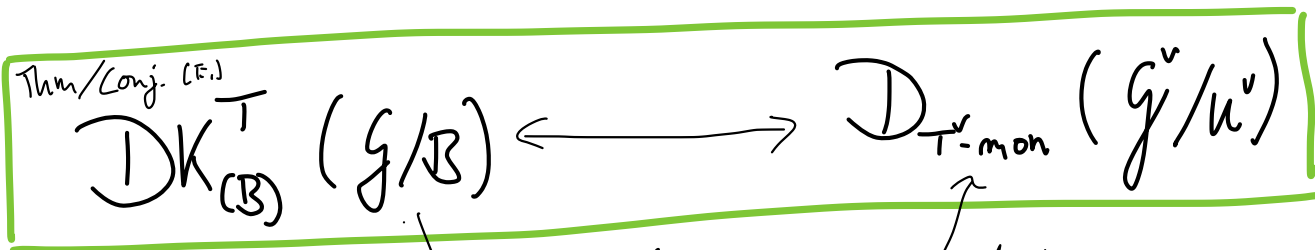
$$k^b(\mathcal{R}\text{-S Bim})$$

k -theory Soergel bimodules

etc . . .

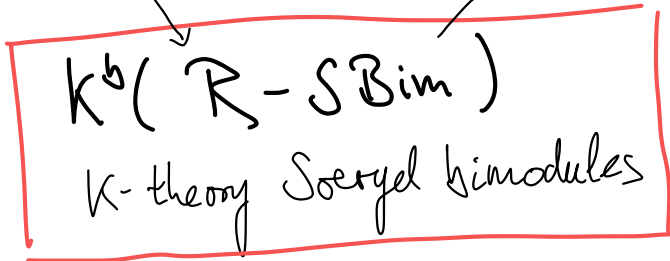
VI Work in progress

THANK YOU



weight complex
functor

realisation
functor



etc . . .

rk in progress

THANK YOU

$$\mathbb{T}(\mathfrak{g}/\mathfrak{B}) \longleftrightarrow \mathcal{D}_{T\text{-mon}}(\mathfrak{g}^{\vee}/\mathfrak{u}^{\vee})$$

weight complex
functor

realisation
functor

$K^b(\mathbb{R}\text{-SBim})$
K-theory Soergel bimodules

etc

gress

$$\left(\frac{v}{u^v} \right)$$

THANK YOU

realisation
functor

$$\begin{matrix} k^b \\ k-t \end{matrix}$$

lt complex
vector

THANK you

The text "THANK you" is written in a black, hand-drawn, cursive style. The word "THANK" is in all caps, while "you" is in lowercase. The text is enclosed within a decorative border composed of alternating red and green dashed lines. The border is roughly rectangular with rounded corners, following the shape of the text.