K-Motives and Local Langlands

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Philosophy/Motivation

Geometric representation theory:

$\{$ representations of algebras $\}$ $\xleftrightarrow{1:1}$ $\{$ sheaves on a space $\}$

strong theorems from AG

Applications

- Compute characters ($\rightsquigarrow$ Kazhdan-Lusztig conjectures)
- Parameterize irreducibles ($\rightsquigarrow$ Langlands parameters)

⋮

Goal of this talk

Theorem (E.)

$$D^b(H_{aff}\text{-mod}) \xleftrightarrow{\sim} DK_r^{Spr}(\mathcal{N}/(G\times\mathbb{G}_m))$$

affine Hecke algebra

K-theoretic sheaves

nilpotent cone

⊂ Categorical unramified local Langlands correspondence

Outline

(1) Affine Hecke algebra

(2) Springer theory

(3) Equivariant K-motives

(4) Results and Outlook

(1) Affine Hecke algebra

p-adic groups

$$\begin{tikzcd}
F \arrow[r, phantom, "\supset"] & \mathcal{O} \arrow[r, two heads] & \kappa = \mathcal{O}/\pi \\
\text{non Archimedean local field} \arrow[u] & \text{ring of integers} \arrow[u] & \text{residue field} \arrow[u]
\end{tikzcd}$$

$$\begin{tikzcd}
\check{G}(F) \arrow[r, phantom, "\supset"] & \check{G}(\mathcal{O}) \arrow[r, two heads] \arrow[d, phantom, "\cup"] & \check{G}(\kappa) \arrow[d, phantom, "\cup"] \\
\text{split reductive group} \arrow[u] & \check{I} \arrow[r, two heads] & \check{B} \\
 & \text{Iwahori} \arrow[u] & \text{Borel} \arrow[u]
\end{tikzcd}$$

Example:

$$\mathbb{Q}_p \supset \mathbb{Z}_p \twoheadrightarrow \mathbb{F}_p = \mathbb{Z}_p/p$$

$$\begin{tikzcd}
GL_n(\mathbb{Q}_p) \arrow[r, phantom, "\supset"] & GL_n(\mathbb{Z}_p) \arrow[r, two heads] \arrow[d, phantom, "\cup"] & GL_n(\mathbb{F}_p) \arrow[d, phantom, "\cup"] \\
 & \left\{\begin{pmatrix} \mathbb{Z}_p^\times & \mathbb{Z}_p & \mathbb{Z}_p \\ p\mathbb{Z}_p & \ddots & \mathbb{Z}_p \\ p\mathbb{Z}_p & p\mathbb{Z}_p & \mathbb{Z}_p^\times \end{pmatrix}\right\} \arrow[r, two heads] & \left\{\begin{pmatrix} \mathbb{F}_p^\times & \mathbb{F}_p & \mathbb{F}_p \\ 0 & \ddots & \mathbb{F}_p \\ 0 & 0 & \mathbb{F}_p^\times \end{pmatrix}\right\}
\end{tikzcd}$$

(1) Affine Hecke algebra

Iwahori-Hecke algebra

$$H(\check{G}(F), \check{I}) = C_c(\check{I} \backslash \check{G}(F) / \check{I}, \mathbb{C})$$

$$\Downarrow$$

$$V^{\check{I}} \longleftarrow V$$

$$H(\check{G}(F), \check{I})\text{-mod} \overset{\sim}{\longleftrightarrow} \mathrm{Rep}^{\check{I}}(\check{G}(F))$$

smooth rep's V
generated by $V^{\check{I}}$
"principal block"

(1) Affine Hecke algebra

Affine Weyl group:

$\check{G} \supset \check{T}$ (max. torus) $\quad Y(\check{T}) = \mathrm{Hom}_{alg.\,grp.}(\mathbb{G}_m, \check{T})$ (cocharacters)

finite Weyl group

$$W_{fin} = N_{\check{G}(F)}(\check{T}(F)) / \check{T}(F)$$

$$Y(\check{T}) \xrightarrow{\sim} \check{T}(F)/\check{T}(\mathcal{O})$$

$$W_{aff} = N_{\check{G}(F)}(\check{T}(F)) / \check{T}(\mathcal{O}) \cong W_{fin} \ltimes Y(\check{T})$$

affine Weyl group

Example:

$GL_n \supset \mathrm{Diag}_n, \quad Y = \mathbb{Z}^n$

$$W_{fin} = \{\text{monomial}\}/\{\text{diagonal}\} = S_n$$

$$(a_i) \mapsto \begin{pmatrix} p^{a_1} & & \\ & \ddots & \\ & & p^{a_n} \end{pmatrix}$$

$$W_{aff} \cong S_n \ltimes \mathbb{Z}^n$$

(1) Affine Hecke algebra

Bruhat decomposition

$$\check{G}(F) = \bigsqcup_{w \in W_{aff}} \check{I} w \check{I}$$

(Gauss + Smith normal form)

$$\mathcal{H}(\check{G}(F), \check{I}) = \bigoplus_{w \in W_{aff}} \mathbb{C}\delta_w$$

indicator on $\check{I} \backslash \check{I} w \check{I} / \check{I}$

Make generic:

$$H_{aff} = \bigoplus_{w \in W_{aff}} \mathbb{Z}[q^{\pm 1}] T_w \longrightarrow$$ Algebra with explicit generators/relations

affine Hecke algebra

$$H_{aff} \underset{\mathbb{Z}[q^{\pm 1}]}{\overset{q \mapsto |\kappa|}{\otimes}} \mathbb{C} \xrightarrow{\sim} \mathcal{H}(\check{G}(F), \check{I})$$

$$T_w \longmapsto \delta_w$$

$\boxed{\text{Study } H_{aff}\text{-mod!}}$

(2) Springer theory

Kazhdan-Lusztig + Ginzburg $\rightsquigarrow$ $\boxed{\text{different realization of } H_{aff}}$

Springer resolution:

$G \supset B \supset T$ over $\mathbb{C}$

G: Langlands dual group to $\check{G}$; B: Borel; T: max. torus

$GL_n \supset \left\{\begin{pmatrix} \ast & \ast \\ 0 & \ast \end{pmatrix}\right\} \supset \left\{\begin{pmatrix} \ast & 0 \\ 0 & \ast \end{pmatrix}\right\}$

$$\mathcal{N} = \{ N \in \mathfrak{g} \mid N \text{ nilpotent} \}$$

$\uparrow \mu$ Springer resolution

$$\tilde{\mathcal{N}} = T^* G/B = G \overset{B}{\times} \mathfrak{n}$$

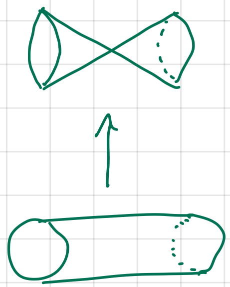
$\circlearrowleft$ $G \times \mathbb{G}_m$ - action

$$\{ A \in \mathbb{C}^{n\times n} \mid A^n = 0 \}$$

$$\uparrow$$

$$\{ (A, F^\bullet) \in \mathbb{C}^{n\times n} \times Fl(\mathbb{C}^n) \mid A F^i \subset F^{i-1} \}$$

n=2:



(2) Springer theory

$$Z = \tilde{\mathcal{N}} \times_{\mathcal{N}} \tilde{\mathcal{N}} \ \overset{\text{diagonal}}{\circlearrowleft} \ G \times \mathbb{G}_m$$

Z: Steinberg variety

Theorem (Kazhdan-Lusztig, Ginzburg)

$$\Big(\underbrace{G_0}_{\substack{G\text{-theory} \\ = K_0(Coh(-))}}\big(Z/G\times\mathbb{G}_m\big), \underbrace{*}_{\text{Convolution product}}\Big) \cong H_{aff}$$

Intuition: $H_{aff} \approx \mathbb{Z}[q^{\pm 1}][\underbrace{W_{fin} \ltimes Y(\check{T})}_{= W_{aff}}]$

$$Z = \bigcup_{w \in W_{fin}} T_{Y_w} Y \quad , \quad Y = (G/B)^2 \qquad Y_w = G\cdot(B/B, wB/B)$$

$$Y_e/G\times\mathbb{G}_m \simeq B\backslash G/G\times\mathbb{G}_m \simeq */T\times\mathbb{G}_m \qquad G_0(*/T\times\mathbb{G}_m) = \underbrace{\mathbb{Z}[q^{\pm1}]}_{G_0(*/\mathbb{G}_m)}[X(T)] = \mathbb{Z}[q^{\pm1}][Y(\check{T})]$$

Turn into equivalence of categories!!

(3) Equivariant K-motives

Sheaves and Cohomology

X CW-complex $\longmapsto$ $D(X)$ $\rightsquigarrow$

derived category of sheaves on X

$$\mathrm{Hom}_{D(X)}(\underline{\mathbb{Z}}, \underline{\mathbb{Z}}[n]) = H^n_{sing}(X, \mathbb{Z})$$

constant sheaf — singular cohomology

$$\mathrm{Hom}_{D(X)}(\mathbb{Z}, \omega_X[n]) = H^{BM}_{-n}(X, \mathbb{Z})$$

Borel-Moore homology

X variety / stack $\longmapsto$?? $\rightsquigarrow$

$$?? = K_n(X)$$

alg. K-theory

$$?? = G_n(X)$$

alg. G-theory

(3) Equivariant K-motives

Definition: $\mathcal{X}$ "nice" stack, e.g. $\mathcal{X} = [X/GL_n]$

$$DK(\mathcal{X}) = \mathrm{Mod}_{KGL_{\mathcal{X}}}(SH(\mathcal{X}))$$

K-motives

spectrum representing alg. K-th.

motivic stable homotopy category

Then:

$$\mathrm{Hom}_{DK(\mathcal{X})}(\mathbb{Z}, \mathbb{Z}[n]) = K_{-n}(\mathcal{X}) \qquad (\text{if } \mathcal{X} \text{ is smooth})$$

$$\mathrm{Hom}_{DK(\mathcal{X})}(\mathbb{Z}, \omega_{\mathcal{X}}[n]) = G_{-n}(\mathcal{X})$$

(3) Equivariant K-motives

Reduced K-motives:

For us: higher K-theory $G_n(Z/g \times g_m)$, $K_n(\mathbb{C})$ $n>0$ is irrelevant.

Definition:

Lurie tensor product

$$DK_r(\mathcal{X}) = DK(\mathcal{X}) \underset{DKT(\operatorname{Spec}\mathbb{C})}{\otimes} D^b(\mathrm{Ab}) = \text{"} DK_r(\mathcal{X}) \underset{K(\mathbb{C})}{\otimes} K(\mathbb{C})/K_{>0}(\mathbb{C}) \text{"}$$

reduced K-motives

Theorem: (E.-Scholbach) DK_r has six operations $f^*, f_*, f_!, f^!, \otimes, \mathrm{Hom}$

(4) Results and Outlook

K-motivic Springer theory:

Let $\mu: \mathcal{X} \to \mathcal{Y}$ be a map of "nice" stacks, such that

(1) $\mathcal{X}$ is regular, μ is proper

(2) $\mathcal{Z} = \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ is cellular (= built from BG's, $\mathbb{A}^n$'s ..)

Let $H(\mathcal{Z}) = (G_0(\mathcal{Z}), *)$, $DK_r^{Spr}(\mathcal{Y}) = \langle \mu_! \mathbb{1} \rangle_{stable} \subset DK(\mathcal{Y})$.

Theorem: (E.):

$$D_{perf}(H(\mathcal{Z})) \xleftrightarrow{\sim} DK_r^{Spr}(\mathcal{Y})$$

(4) Results and Outlook

Corollary: (⇝ Introduction)

$$D^b(\mathcal{H}_{\mathrm{aff}}) \xleftrightarrow{\sim} DK_r^{\mathrm{spa}}(\mathcal{N}/\mathfrak{g}\times\mathbb{G}_m)$$

Corollary:

$$D^b(\mathrm{Rep}^{\check{I}}(\check{\mathfrak{g}})) \xleftrightarrow{\sim} DK_r^{\mathrm{spa}}(\mathcal{N}/\mathfrak{g}\times\mathbb{G}_m)_{q=|K|}$$

+ similar results for $DM_r^{\mathrm{spa}}(\mathcal{N}/\mathfrak{g}\times\mathbb{G}_m)$, $D^{\mathrm{spa}}(\mathcal{N}/\mathfrak{g}\times\mathbb{G}_m)$, ...

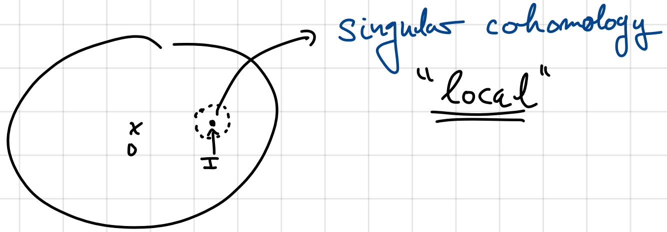
(⇝ E.-Stroppel, Rider)

(4) Results and Outlook

"Globalization": Atiyah-Segal completion

$$I = \ker\left(K_0(\cdot/G) \longrightarrow K_0(pt)\right)$$

$$K_0^{top}(X/G)^{\wedge}_{\mathbb{Q},I} \cong \prod_n H^n_{sing}(X/G, \mathbb{Q})$$



$\mathrm{Spec}(K_0(\cdot/G))$

Obtain specialization functors:

$$\begin{array}{ccccc} DK(\mathfrak{X})_{\mathbb{Q}} & \longleftarrow & DM(\mathfrak{X})_{\mathbb{Q}} & \longrightarrow & D(\mathfrak{X}(\mathbb{C}))_{\mathbb{Q}} \\ \uparrow & & & & \uparrow \\ \text{global} & & & & \text{local around } I. \end{array}$$

$\rightsquigarrow$ Deligne-Langlands correspondence for H_{aff}

(4) Results and Outlook

Categorified Chern character:

$$\mathcal{L}\mathcal{X} = \mathcal{X} \times_{\mathcal{X}\times\mathcal{X}} \mathcal{X} \qquad \text{Loop space}$$

Conj: There is a categorified Chern character functor ch, such that the following diagram commutes:

$$\begin{tikzcd}
\mathrm{DK}^{Spr}(\mathcal{N}/G\times\mathbb{G}_m) \arrow[r, "ch"] \arrow[d, "\wr"] & \mathrm{Coh}^{Spr}(\mathcal{L}(\mathcal{N}/G\times\mathbb{G}_m)) \arrow[d, "\wr"] \\
D_{perf}(H_{aff}) \arrow[r, equal] & D_{perf}(H_{aff})
\end{tikzcd}$$

Coherent Springer theory
Ben-Zvi–Chen–Helm–Nadler, Zhu
Hellmann, Hemo, ...

Geometric Langlands

$$\begin{tikzcd}
\text{Constructible} \arrow[rr, dash] \arrow[dr, dash] & & \text{Coherent} \arrow[dl, dash] \\
& \text{K-motivic} &
\end{tikzcd}$$

(4) Results and Outlook

Further:

Theorem (E., Taylor) "Global Koszul duality"

$$DK_r(B\backslash G/B) \xleftrightarrow{\sim} D_{mon}(\check{U}\backslash \check{G}/\check{U})$$

Conj. "Quantum K-theoretic Satake"

$$DK_r(G(O)\times \mathbb{G}_m \backslash G^{(F)}/G(F)) \xleftrightarrow{\sim} D_{u_q(\check{g})}(O_q(\check{G})\text{-mod})$$

$$\vdots$$

ありがとうございます 💖