

# WEDNESDAY SEMINAR WS16/17: $SL_2$

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## 1. INTRODUCTION

The special linear group  $SL_2$  with all its incarnations has an appearance in many basic—yet often theory shaping—examples from different areas of mathematics. We want to explore some of them in this seminar.

We devote a big part of our time to  $SL_2(\mathbb{R})$ . We will study the interplay of representation theory, harmonic analysis and differential geometry on this group and apply it to the theory of modular and automorphic forms (see 2.1 and 2.4). On our way, we will also see the significance of  $SL_2(\mathbb{R})$  and  $SL_2(\mathbb{C})$  in the theory of elliptic curves and the classification of hyperbolic 2- and 3-manifolds (see 2.3).

The isomorphisms  $SL_2(\mathbb{R}) \cong \text{Spin}(2,1)$  and  $SL_2(\mathbb{C}) \cong \text{Spin}(3,1)$  allow us to interpret elements of  $SL_2$  as symmetries of spacetime. We use this to see how certain representations are related to elementary particles (see 2.2).

In the end, we showcase some ideas from geometric representation theory. We study how the Borel–Weil–Bott theorem and the Beilinson–Bernstein localisation theorem connect the geometry of  $\mathbb{CP}^1$  to the representation theory of  $SL_2(\mathbb{R})$  (see 2.5). After this, we analyse how representation theory changes when we pass to  $SL_2(\mathbb{F}_q)$  (see 2.6).

The programme in its broad selection of topics is not designed to prove a big theorem or to get deeply immersed in one specific theory. But it will hopefully be able to fulfil two goals. Firstly, a variety of introductory talks about topics even seemingly unrelated to  $SL_2$  are meant to increase our general mathematical knowledge. Secondly, its more advanced talks should then draw in broad strokes connections in between those topics and our research training group in general.

Furthermore, the programme is subdivided in mostly independent short parts of two to four talks. Please, feel free to talk to other people in your part and move stuff around a bit if needed.

## 2. PROGRAMME

### 2.1. Representation Theory and Harmonic Analysis.

**Talk 1: Warm-up.** (19.10., ?) This talk is meant to make us all comfortable with  $SL_2$  in an elementary fashion.

Discuss all the different incarnations of  $SL_2(R)$ . Make sure to explain that we sometimes (also depending on the ring  $R$ ) see it as an algebraic group, a topological group, a Lie group – or merely as an abstract group. Do this in a clean language: talk about group objects in categories.

Quickly recapitulate the Lie algebra  $\mathfrak{sl}_2$ , don't spend too much time here. Semisimple groups and Lie algebras carry many different interesting substructures and decompositions. Explain some of them on the example of  $SL_2$ . You should include the Cartan, Bruhat and Iwasawa decompositions and perhaps also explain the Jordan-Chevalley decomposition of elements in a semisimple and unipotent part. Be sure to shortly mention the general statements as well.

The second half of your talk should concern representation theory. Again start with distinguishing the different meanings of representations, algebraic, analytic, smooth, etc.

Explain Weyl's unitary trick and how to identify finite dimensional

- representations of  $\mathfrak{sl}_2(\mathbb{C})$ ,
- smooth representations of  $\mathrm{SL}_2(\mathbb{R})$ ,
- algebraic/holomorphic representations of  $\mathrm{SL}_2(\mathbb{C})$  and
- unitary representations of  $\mathrm{SU}(2)$ .

Classify the finite dimensional representations of  $\mathfrak{sl}_2(\mathbb{C})$ . Be sure to draw them as diagrams.

As an application, mention the Lefschetz decomposition of the cohomology of a Kähler manifold.

If you have time left explain the finite dimensional *smooth* representations of  $\mathrm{SL}_2(\mathbb{C})$ .

References: are abundant here. A nice list of decompositions can be found here. For the unitary trick see [Kna01].

**Talk 2: Harish-Chandra modules.** (26.10., ?) In this talk we transition to the case of infinite dimensional representations on the example of  $\mathrm{SL}_2(\mathbb{R})$ .

Explain what one could mean by an infinite dimensional representation: An action on a Banach space, Hilbert space, a locally convex topological vector space, etc.

Explain the following dilemma. The action of a group  $G$  on a space  $X$  can induce many non-equivalent representations on all the different choices of function spaces on  $X$ .

Define the notion of infinitesimal equivalence and Harish-Chandra modules, and explain how this can circumvent the dilemma. Perhaps use the example of  $S^1 = \mathrm{SO}(2)$ .

Classify the irreducible admissible  $(\mathfrak{sl}_2(\mathbb{R}), \mathrm{SO}(2))$ -modules and discuss which of them are unitary. Please, replace as many formulas by diagrams as possible.

Reference: [Cas] is a really nice essay. If you do the classification of irreducible  $(\mathfrak{g}, K)$ -modules from Section 3, please draw the diagrams from Section 8. For the classification of the unitary representations see Chapter III in [HT92].

**Talk 3: The principal and discrete series.** (02.11., Hannah Bergner) Here we want to prove that both the principal and discrete series, we have seen as Harish-Chandra modules in the last talk, really come from representations of  $G$ .

Define the principal series as the space  $\mathrm{Ind}^\infty(\chi)$  for a character  $\chi : B \rightarrow \mathbb{C}^\star$ , for  $B \subset \mathrm{SL}_2(\mathbb{R})$  the group of upper triangular matrices.

Explain how  $\mathrm{Ind}^\infty(\chi)$  is dual to  $\mathrm{Ind}^\infty(\chi^{-1})$ .

Explain how  $\mathfrak{sl}(\mathbb{R})$ ,  $\mathrm{SO}(2)$  and the Casimir element acts.

Discuss when they are reducible or irreducible, and how all the Harish-Chandra modules from the last talk appear as subquotients.

Discuss which of them are unitary.

The rest of this talk should be devoted to the discrete series. Try to answer the following questions:

- How can we realize the discrete series in terms of functions on the group or sections in line bundles on the upper half plane  $\mathcal{H}$ .
- What is the holomorphic and antiholomorphic discrete series. Make sense of the translation *holomorphic*  $\rightleftharpoons$  *highest weight vector*.

This talk is closely related to the preceding one. Make sure to coordinate your talks.

Reference: Again Casselmann's essay [Cas]. In particular: Section 8, 19, 20, 21.

**Talk 4: Harmonic Analysis.** (09.11., Nelvis Fornasin) This talk consists of three increasingly difficult parts. The goal is to understand the connection between harmonic analysis and representation theory. Firstly, we want to see the relation between Fourier theory and Pontryagin duality for locally compact abelian groups. Secondly, we want to understand the Peter–Weyl theorem for compact Lie-groups as a generalization. And lastly, we want to understand what can be rescued in the case of non-compact semisimple groups like  $SL_2(\mathbb{R})$ . The big goal is to decompose  $L^2(SL_2(\mathbb{R}))$  in terms of the representations of the last two talks.

Quickly recapitulate the classical Fourier theory for functions on  $S^1$  and  $\mathbb{R}$ . Explain Pontryagin duality and the Fourier transform as an isomorphism  $L^2(G) \cong L^2(\hat{G})$ .

Introduce matrix coefficients and the Peter–Weyl theorem for compact groups. Discuss the case  $K = SO(2)$ .

Now for the hard part. It is impossible to treat all the material necessary for the Plancherel formula for  $SL_2(\mathbb{R})$  in detail. The following points should be mentioned:

- What is a Plancherel formula in general? What is a Plancherel measure for the unitary dual?
- Recall the unitary dual of  $SL_2(\mathbb{R})$ , and introduce the Plancherel measure here.
- State the Plancherel formula for  $SL_2(\mathbb{R})$ . Sketch the proof.
- Justify why one differentiates between the continuous principal and the discrete series.
- Explain how this can be stated abstractly in a decomposition of the space  $L^2(SL_2(\mathbb{R}))$ .

Reference: Varadarajan’s [Var89] is a comprehensive source for everything needed in this talk (and much more). Relevant for us are: Sections 1.6, 2.1,2.2,4.1, the introduction of 6.1 and 6.6. You can also use [Wal73].

## 2.2. Applications in Physics.

**Talk 1: The Poincaré Group and Elementary Particles.** (16.11., Jørgen Lye)

## 2.3. 2- and 3-manifolds, Fuchsian and Kleinian groups.

**Talk 1: Riemann surfaces.** (23.11., Martin Schwald) The classification of Riemann surfaces is deeply connected to  $SL_2$ . In this talk we want to understand the connection in the case of elliptic curves.

Talk about the goal: We want to understand the connection between  $SL_2$  and Riemann surfaces.

Quickly recapitulate the statement of the uniformization theorem: Every simply connected Riemann surface is either  $\mathbb{C}P^1$ ,  $\mathbb{C}$  or  $\mathcal{H}$  (trinity). Discuss the implications for Riemann surfaces which are not necessarily simply connected.

We hence want to understand quotients!

The rest of the talk is devoted to elliptic curves:

Give the definition, and the different forms in which an elliptic curve can be expressed.

Describe how different moduli spaces for elliptic curves can be constructed as double quotients  $SO(2) \backslash SL_2(\mathbb{R}) / \Gamma$ .

Reference: For example [Kob93], but you can use whatever you want.

**Talk 2: Hyperbolic 2- and 3-manifolds.** (30.11., René Recktenwald) In this talk we want to treat the hyperbolic case. For this we need a good general understanding of the upper half plane and its hyperbolic metric. We then need to discuss Fuchsian groups (discrete subgroups of  $PSL_2(\mathbb{R})$ ) with an emphasis on the modular

group  $\mathrm{PSL}_2(\mathbb{Z})$  and the relation to hyperbolic Riemann surfaces. In a survey-style fashion we then want to understand how the picture changes when passing to dimension 3, and how hyperbolic 3-manifolds come as quotients of  $\mathrm{SL}_2(\mathbb{C})$ . Reference: [Kat92].

#### 2.4. Number Theory.

**Talk 1: Modular and automorphic forms.** (07.12., Emanuel Scheidegger) This talk should give a short and motivated introduction to modular forms and their friends and explain their connection to elliptic curves and representation theory.

What is a modular form? In which ways can we express a modular form?

Examples of modular forms and connection to elliptic curves.

What are the generalizations. In particular, what do we mean by an automorphic form.

How can we understand a modular/automorphic form as a section of a line bundles on quotients of  $\mathrm{SL}_2(\mathbb{R})$ ?

How do properties translate, as for example holomorphic  $\leftrightarrow$  highest weight vector?

It would be desirable to understand a sentence of the form "A modular form (cusp form of weight  $\geq 2$ ) is a highest weight vector of a discrete series summand of  $L^2(\mathrm{SL}_2(\mathbb{Z}) \backslash \mathrm{SL}_2(\mathbb{R}))$ ".

References: [Kob93] for modular forms and elliptic curves. For the connection to representation theory [Bum97], and for a nice informal account, just concerned with modular forms, see here [Ven].

**Talk 2: Tamagawa number.** (14.12., Max Schmidtke) This talk should introduce the concept of adèles and as an example compute the Tamagawa number of  $\mathrm{SL}_2(\mathbb{A})/\mathrm{SL}_2(\mathbb{Q})$ .

**Talk 3: Hecke operators.** (11.01., Fritz Hörmann) This should be a survey-style talk about further directions.

What are Hecke operators and Hecke eigenforms.

How does looking at  $\mathrm{SL}_2(\mathbb{A})$  help here?

What are automorphic functions on  $\mathrm{SL}_2(\mathbb{A})$ ?

Some words about the Langlands program.

Reference: [GGPS90], [Bum97].

#### 2.5. Geometric Representation Theory.

**Talk 1: Geometric construction of finite dimensional representations.** (18.01., Elmiro Vetere) We return to the first talk of this seminar, namely finite dimensional representations. This time we want to take a more algebro-geometric viewpoint. The goal of this talk is to show that the finite dimensional irreducible algebraic representations of  $\mathrm{SL}_2(\mathbb{C})$  can be constructed as the cohomology groups  $H^0(\mathbb{P}_{\mathbb{C}}^1, \mathcal{O}(n))$  or dually  $H^1(\mathbb{P}_{\mathbb{C}}^1, \mathcal{O}(-n))$ . This is the statement of the Borel–Weyl–Bott theorem. This should be done as explicitly as possible. So answer the questions: What are the line bundles on  $\mathbb{P}^1$ ? Why are they  $G$ -equivariant, how can they be constructed from characters of  $T$ ? Why are their global sections just homogeneous polynomials of a certain fixed degree? What is Serre duality doing in this baby case? The second half of the talk should be devoted to the Weyl character formula. What do we mean by a character here? What are they for the modules just constructed? Sketch how the general character formula for arbitrary semisimple algebraic groups can be reduced to  $\mathrm{SL}_2$ .

References: For example this. For an algebraic proof of the Weyl character formula: [Spr68].

*Talk 2: D-modules on  $\mathbb{P}_{\mathbb{C}}^1$ .* (25.01.,?) In the talks in the beginning of the semester, we found geometric realizations for the  $(\mathfrak{sl}_2(\mathbb{C}), SO(2))$ -Harish-Chandra modules as the  $SO(2)$ -finite vectors in spaces of sections in line bundles on quotients of  $SL_2(\mathbb{R})$ . This talk presents a different approach. Starting from the observation that the ring of global differential operators on the flag variety  $SL_2(\mathbb{C})/B = \mathbb{P}_{\mathbb{C}}^1$  is isomorphic to the  $U(\mathfrak{sl}_2(\mathbb{C}))/Z$ , universal enveloping algebra of  $\mathfrak{sl}_2(\mathbb{C})$  modulo its center, we want to understand how categories of D-modules on  $\mathbb{P}_{\mathbb{C}}^1$  are equivalent to categories of Harish-Chandra modules.

Reference: There are some general references on D-modules, but you would need to work out parts of the specific example on your own.

## 2.6. $SL_2$ over a finite field.

**Talk 1: Defining Characteristic.** (01.02., Giovanni Zaccanelli) In this talk we want to understand the representations of  $SL_2(\mathbb{F}_q)$  in *defining characteristic*, i.e. representations in characteristic  $p$ . The main idea is to instead study rational representations of the algebraic group  $SL_2/\overline{\mathbb{F}}_p$ . Here one first proceeds as in the characteristic 0 case, namely uses sections of line bundles on  $\mathbb{P}^1/\overline{\mathbb{F}}_p$  (homogeneous polynomials in two variables of a set degree). But these representations are not longer irreducible! Here the Steinberg tensor-product theorem helps to understand the situation. Then one restricts them to  $SL_2(\mathbb{F}_q)$ , and tries to understand if irreducibles stay irreducible (Steinberg restriction theorem).

References: This is worked out very explicitly and elementary in [Bon11] Section 10.1. If you have more time you can consult [Jan03] for the generalizations to all semisimple algebraic groups over  $\overline{\mathbb{F}}_p$ .

**Talk 2: Characteristic Zero.** (08.02.,?) Here we want to understand the representation theory of  $SL_2(\mathbb{F}_q)$  in characteristic zero and try to get formulas for the character tables for all  $q$ . Do whatever you like best.

References: This is again done explicitly in [Bon11]. A nice source is also [Hum75].

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