Soergel and Springer

Motives and Correspondences

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Today's talk: Context

Philosophy:

$$\left\{\text{representations of } \begin{matrix}\text{group } G\\ \text{Lie alg. } \mathfrak{g}\\ \text{algebra } A\end{matrix}\right\} \longleftrightarrow \left\{\text{geom. objects on space } X\right\}$$

Applications:

- parameterize representations (Langlands parameter)
- calculate characters (Kazhdan-Lusztig conj.)
- find "canonical" bases (Kashiwara, Lusztig)

Today :

$$\{\text{representations of convolution algebra}\} \xleftrightarrow{1:1} \{\text{Springer motives on some space } S\}$$

1. Convolution algebras
2. Chow motives / Voevodsky motives
3. Weight structures
4. Motivic Springer theory

Toy Example: Convolution for functions

Basic operations:

X finite set $\rightsquigarrow$ $\mathbb{C}^X = \{\alpha : X \to \mathbb{C}\}$ functions on X.

$f : X \to Y$ map $\rightsquigarrow$ $f^* : \mathbb{C}^Y \to \mathbb{C}^X$, $f^*(\alpha) = \alpha \circ f$ pullback

$f_! : \mathbb{C}^X \to \mathbb{C}^Y$, $(f_!(\alpha))(y) = \sum_{x \in f^{-1}(\{x\})} \alpha(x)$ pushforward

$\cap : \mathbb{C}^X \times \mathbb{C}^X \to \mathbb{C}^X$, $(\alpha \cap \beta)(x) = \alpha(x) \cdot \beta(x)$

Convolution product:

X, Y, Z finite sets

$$* : \mathbb{C}^{X\times Y} \times \mathbb{C}^{Y\times Z} \longrightarrow \mathbb{C}^{X\times Z}$$

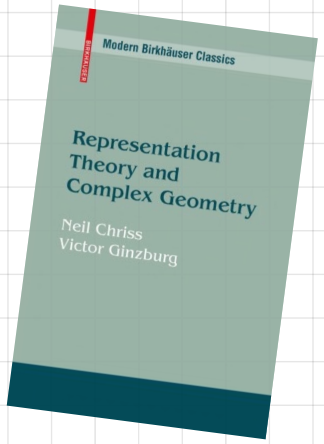
$$\alpha * \beta = \pi_! \Delta^*\big(p^*(\alpha) \cap q^*(\beta)\big)$$

$$\begin{tikzcd}
\alpha & & \beta & \\
X\times Y & & Y\times Z & \\
& X\times Y\times Y\times Z \arrow[ul, "p"] \arrow[ur, "q"'] & & p^*(\alpha)\cap q^*(\beta) \\
& X\times Y\times Z \arrow[u, "\Delta"'] \arrow[d, "\pi"] & & \Delta^*\big(p^*(\alpha)\cap q^*(\beta)\big) \\
& X\times Z & & \pi_!\Delta^*\big(p^*(\alpha)\cap q^*(\beta)\big)
\end{tikzcd}$$

$$(\alpha*\beta)(x,z) = \sum_{y\in Y} \alpha(x,y)\cdot\beta(y,z) \quad \overset{X=Y=Z}{\rightsquigarrow} \quad (\mathbb{C}^{X\times X}, *) \cong (\mathbb{C}^{n\times n}, \cdot) \qquad n = |X|$$

$\uparrow$ Matrix multiplication!

Convolution for cohomology classes


Modern Birkhäuser Classics
Representation Theory and Complex Geometry
Neil Chriss
Victor Ginzburg

(Co-)homology:

$$X \begin{cases} \text{manifold} \\ \text{variety} \\ \text{stack} \\ \vdots \end{cases} \rightsquigarrow H_\bullet(X) \begin{cases} \text{singular homology} \\ \text{deRham cohomology} \\ \text{K-theory} \\ \text{Chow groups} \\ \vdots \end{cases} + \text{equivariant versions}$$

$$\rightsquigarrow f^*, f_!, \cap.$$

Convolution product:

$$CH(X \times_S Y) \times CH(Y \times_S Z) \longrightarrow CH(X \times_S Y)$$

$$\alpha * \beta = \pi_! \Delta^* \big(p^*(\alpha) \cap q^*(\beta) \big)$$

$$\begin{tikzcd} X \arrow[dr] & \\ Y \arrow[r] & S \\ Z \arrow[ur] & \end{tikzcd}$$

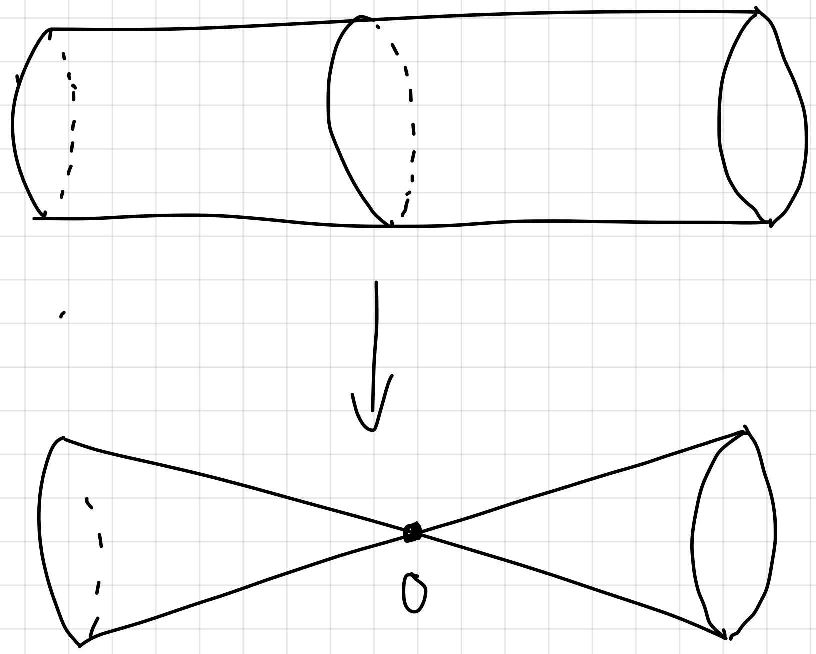
⇝ associative

Symmetric group via convolution

Springer resolution

$$\begin{tikzcd} X = T^*\mathcal{F}\ell_n = \left\{ (A, (0 \subset V_1 \subset V_2 \subset \cdots \subset \mathbb{C}^n)) \;\middle|\; \begin{array}{l} AV_i \subset V_{i-1}, \\ \dim_{\mathbb{C}} V_i = i \end{array} \right\} \arrow[d, "\mu"', "\leftarrow \text{Springer resolution}"] \\ S = \mathcal{N} = \{ A \in \mathbb{C}^{n \times n} \mid A^n = 0 \} \leftarrow \text{nilpotent cone} \end{tikzcd}$$

$n = 2$

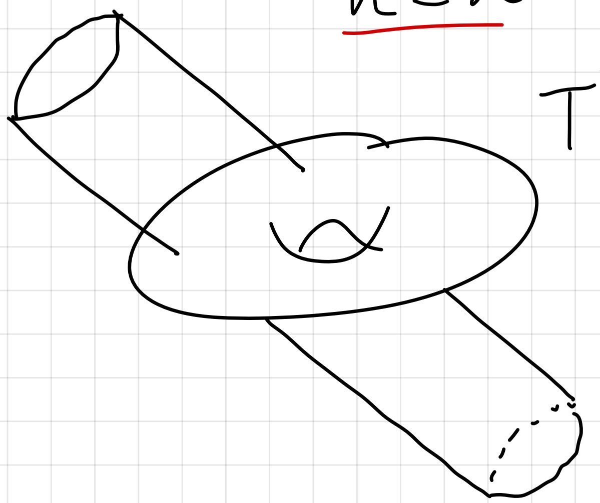


$X \cong T^*\mathbb{P}^1_{\mathbb{C}}$

$S \cong \{ A \in \mathbb{C}^{2\times 2} \mid \det A = \operatorname{tr} A = 0 \}$

$$\begin{tikzcd} (CH^0(\overset{\text{Steinberg variety}}{X \times_S X}), *) \arrow[d, mapsto] & = & \mathbb{C}[S_n] \arrow[d, mapsto] & \text{symmetric group} \\ CH^0(\mu^{-1}(\underset{\text{Springer fibre}}{A_\lambda})) & = & S_\lambda & \text{Specht module} \end{tikzcd}$$

$\underline{n=2}$

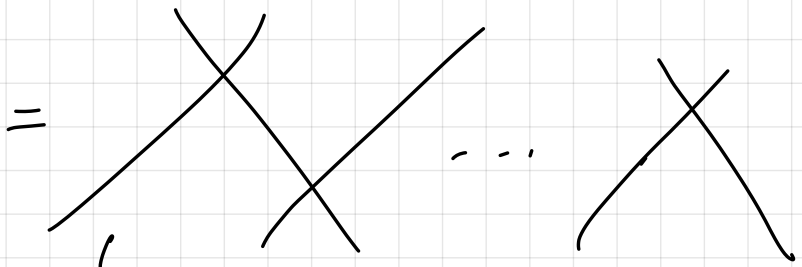


$T^*\mathbb{P}^1 \cup_{\mathbb{P}^1} \mathbb{P}^1 \times \mathbb{P}^1$

$$\mu^{-1}\begin{pmatrix}0&1&&\\&0&1&\\&&\ddots&\ddots\\&&&1\\&&&0\end{pmatrix} = \{*\}$$

$\rightsquigarrow$ trivial rep

$$\mu^{-1}\begin{pmatrix}0&0&&&\\&0&1&&\\&&0&1&\\&&&0&\ddots\\&&&&1\\&&&&0\end{pmatrix} =$$



$\rightsquigarrow$ (reduced) permutation rep.

$$\mu^{-1}(A)$$

$\Uparrow$

extremely intricate

Convolution Algebras in Representation Theory

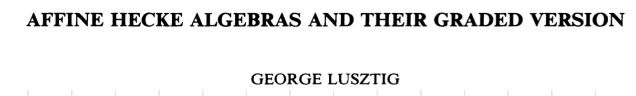
Different choice of homology th. $H_\bullet$ and $\mu : X \to S$ yield:

- Weyl groups $\quad X = T^*G/B \longrightarrow S = \mathcal{N} \quad CH^0 = H^{BM}_{top}$

A Construction of Representations of Weyl Groups

T. A. Springer

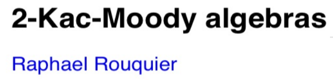
- affine Hecke algebras $\quad X = T^*G/B \longrightarrow S = \mathcal{N} \quad CH^\bullet_{G\times\mathbb{G}_m}$ or $K_0^{G\times\mathbb{G}_m}$



AFFINE HECKE ALGEBRAS AND THEIR GRADED VERSION

GEORGE LUSZTIG

- KLR algebras $\quad X = \bigsqcup_{\underline{d}} \overset{\text{quiver flag variety}}{\overset{\downarrow}{\mathcal{F}\ell_{\underline{d}}}} \longrightarrow S = \mathrm{Rep}(\overset{\text{ADE-quiver}}{\overset{\downarrow}{Q}}) \quad CH^\bullet_{GL(d)\times\mathbb{G}_m} = H^{BM}_{GL(d)\times\mathbb{G}_m}$



2-Kac-Moody algebras

Raphael Rouquier

A diagrammatic approach to categorification of quantum groups I

Mikhail Khovanov, Aaron D. Lauda

Canonical bases and Khovanov-Lauda algebras

M. Varagnolo, E. Vasserot

- ...

- Category $\mathcal{O}_0(\mathfrak{g}^L)$ $\quad X = \bigsqcup_{\underline{w}\in W} \overset{\text{Bott-Samelson resolution}}{\overset{\wr}{BS(\underline{w})}} \to S = G/B \quad CH^\bullet = H^{BM}_\bullet \quad$ "Soergel ∩ Springer"

Convolution for Motives

S variety $/k$

$$\mathrm{Corr}(S) = \begin{cases} \text{Objects}: & M(X/S) \qquad \underset{}{\overset{\text{smooth over } k}{\overset{\downarrow}{X}}} \to S \text{ projective} \\ \text{Morphisms}: & \mathrm{Hom}(M(X/S), M(Y/S)) = CH^{\dim Y}(X \times_S Y)_{\mathbb{Q}} \end{cases}$$

additive structure $\oplus = \uplus$ and $\mathbb{Q}$-linear

$$\mathrm{Chow}_{\mathrm{eff}}(S) = \mathrm{Kar}(\mathrm{Corr}(S))$$

$$M(\mathbb{P}^1_S/S) = \mathbb{Q} \oplus \mathbb{L} \in \mathrm{Chow}_{\mathrm{eff}}(S)$$

$$\mathrm{Chow}(S) = \mathrm{Chow}_{\mathrm{eff}}(S)[\mathbb{L}^{\otimes -1}]$$

↑ Chow motives

Pure to Mixed

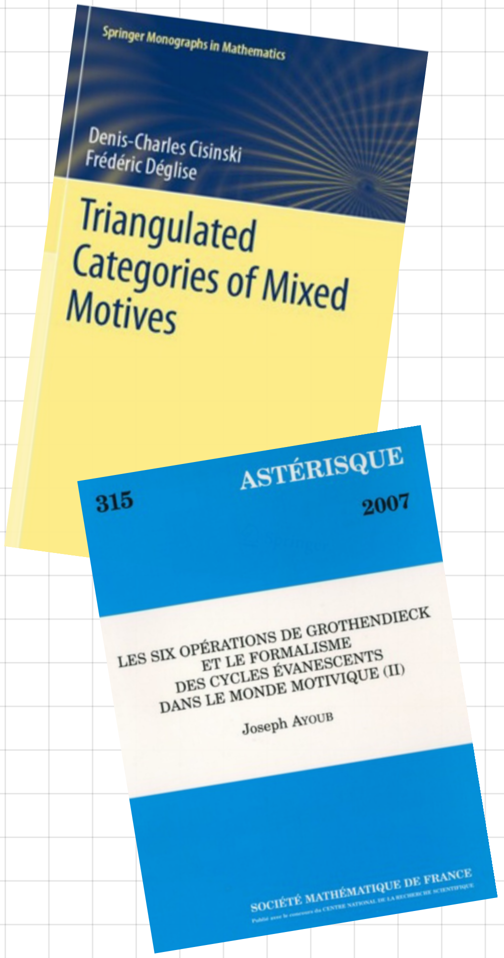
Grothendieck's Chow motives   Voevodsky's mixed motives

$$\begin{array}{ccc} \text{additive} & & \text{triangulated} \\ \downarrow & & \downarrow \\ \mathrm{Chow}(S) & \subset & DM(S) \\ \cup & & \cup \\ M(X/S) & & M(X/S) \\ X \to S & & X \to S \\ \uparrow \quad \uparrow & & \uparrow \\ \text{smooth} \ \ \text{projective} & & \text{arbitrary} \end{array}$$

"pure motives"   "mixed motives"

⇝ heart of weight structure on DM

Properties of Mixed Motives



Springer Monographs in Mathematics

Denis-Charles Cisinski
Frédéric Déglise

Triangulated Categories of Mixed Motives

ASTÉRISQUE
315
2007

LES SIX OPÉRATIONS DE GROTHENDIECK ET LE FORMALISME DES CYCLES ÉVANESCENTS DANS LE MONDE MOTIVIQUE (II)

Joseph Ayoub

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

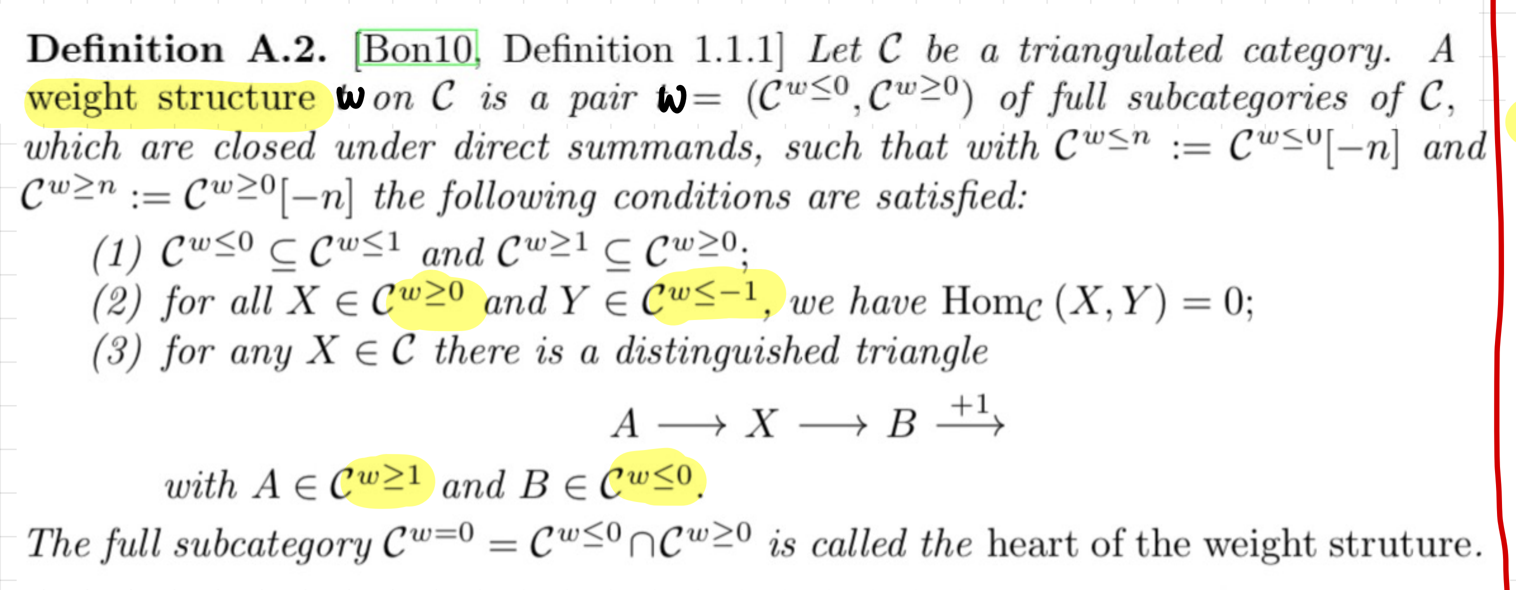
$S \longmapsto DM(S)$ $\mathbb{Q}$-linear $\otimes$-triangulated

$f: S \to T \longmapsto f^*, f_*, f_!, f^!$

- Similar to $D(S^{an}(\mathbb{C}))$ or $D_{ét}(S/\mathbb{F}_p, \mathbb{Q}_\ell)$
  - Localisation triangles, projection formulae, base change, $\mathbb{A}^1$-invariant.
- $M(\mathbb{P}^1_S) = p_* p^* \mathbb{Q} = \mathbb{Q} \oplus \mathbb{Q}(1)[2]$ $(p: \mathbb{P}^1_S \to S)$ $-(1) = - \otimes \mathbb{Q}(1)$ Tate twist
- $\mathrm{Real}_\ell : DM(S) \to D_{ét}(S, \mathbb{Q}_\ell)$ (for $\ell \in \mathcal{O}_S^\times$)
- $\mathrm{Hom}_{DM(S)}(\mathbb{Q}, \mathbb{Q}(p)[q]) = CH^p(S, 2p-q)$ higher Chow group
- $DM(S) \overset{?}{=} D(MM(S))$ $\Longleftrightarrow$ standard conjecture ?motivic t-structure?
- $DM(S)^{w=0} = \mathrm{Chow}(S)$ weight structure

Weight Structures

$$\underset{\substack{\uparrow\\ \text{heart of weight structure}}}{\operatorname{proj}(\mathcal{A})} \subset K^b(\operatorname{proj}(\mathcal{A})) = D^b(\mathcal{A}) \supset \underset{\substack{\uparrow\\ \text{heart of t- structure}}}{\mathcal{A}}$$



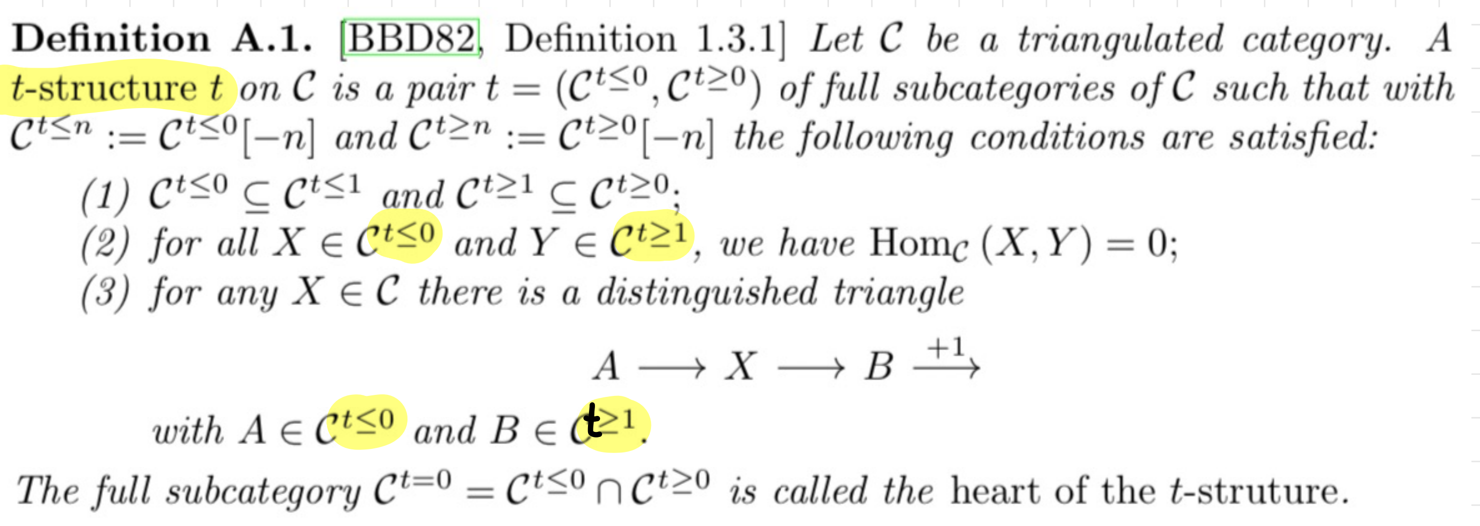
Definition A.2. [Bon10, Definition 1.1.1] *Let $\mathcal{C}$ be a triangulated category. A* weight structure $\boldsymbol{w}$ *on $\mathcal{C}$ is a pair* $\boldsymbol{w} = (\mathcal{C}^{w\leq 0}, \mathcal{C}^{w\geq 0})$ *of full subcategories of $\mathcal{C}$, which are closed under direct summands, such that with $\mathcal{C}^{w\leq n} := \mathcal{C}^{w\leq 0}[-n]$ and $\mathcal{C}^{w\geq n} := \mathcal{C}^{w\geq 0}[-n]$ the following conditions are satisfied:*

(1) $\mathcal{C}^{w\leq 0} \subseteq \mathcal{C}^{w\leq 1}$ and $\mathcal{C}^{w\geq 1} \subseteq \mathcal{C}^{w\geq 0}$;
(2) for all $X \in \mathcal{C}^{w\geq 0}$ and $Y \in \mathcal{C}^{w\leq -1}$, we have $\mathrm{Hom}_{\mathcal{C}}(X,Y) = 0$;
(3) for any $X \in \mathcal{C}$ there is a distinguished triangle

$$A \longrightarrow X \longrightarrow B \xrightarrow{+1}$$

with $A \in \mathcal{C}^{w\geq 1}$ and $B \in \mathcal{C}^{w\leq 0}$.

The full subcategory $\mathcal{C}^{w=0} = \mathcal{C}^{w\leq 0} \cap \mathcal{C}^{w\geq 0}$ is called the heart of the weight struture.



Definition A.1. [BBD82, Definition 1.3.1] *Let $\mathcal{C}$ be a triangulated category. A* t-structure t *on $\mathcal{C}$ is a pair $t = (\mathcal{C}^{t\leq 0}, \mathcal{C}^{t\geq 0})$ of full subcategories of $\mathcal{C}$ such that with $\mathcal{C}^{t\leq n} := \mathcal{C}^{t\leq 0}[-n]$ and $\mathcal{C}^{t\geq n} := \mathcal{C}^{t\geq 0}[-n]$ the following conditions are satisfied:*

(1) $\mathcal{C}^{t\leq 0} \subseteq \mathcal{C}^{t\leq 1}$ and $\mathcal{C}^{t\geq 1} \subseteq \mathcal{C}^{t\geq 0}$;
(2) for all $X \in \mathcal{C}^{t\leq 0}$ and $Y \in \mathcal{C}^{t\geq 1}$, we have $\mathrm{Hom}_{\mathcal{C}}(X,Y) = 0$;
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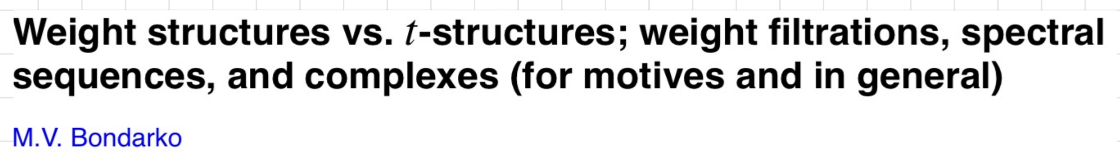
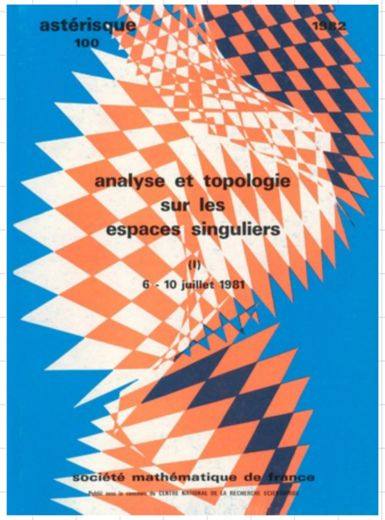
The full subcategory $\mathcal{C}^{t=0} = \mathcal{C}^{t\leq 0} \cap \mathcal{C}^{t\geq 0}$ is called the heart of the t-struture.



Weight structures vs. t-structures; weight filtrations, spectral sequences, and complexes (for motives and in general)

M.V. Bondarko



$$\mathcal{C} \longrightarrow K^b(\mathcal{C}^{w=0}) \qquad\qquad D^b(\mathcal{C}^{t=0}) \longrightarrow \mathcal{C}$$

Bondarko's weight complex functor

Beilinson's realisation functor

⚠ not equivalences (in general)

A Toy Model of Mixed Motives

DM(S) has weight structure, such that

$$DM(S)_{w=0} = Chow(S)$$

$$\rightsquigarrow \quad DM(S) \longrightarrow K^b(Chow(S))$$

$\uparrow$

not an equivalence

BUT

can be an equivalence on certain subcategories

Motivic Springer Theory

$$\mu : \underset{\substack{\uparrow \\ \text{smooth}/\mathbb{F}_q}}{X} \underset{\substack{\uparrow \\ \text{proj.}}}{\longrightarrow} S \quad , \qquad G \text{ equivariant}$$

Def:

$$DM^{Spr}_{\underset{\substack{\uparrow \\ \text{Springer motives}}}{G}}(S) = \langle \mu_!(\mathbb{Q}) = M(X/S) \rangle_{\underset{\substack{\uparrow \\ \text{triang.}}}{\Delta},\ \underset{\substack{\uparrow \\ \text{direct summands}}}{\oplus},\ \underset{\substack{\nwarrow \\ \text{Tate twist}}}{(1)}}$$

$$\underset{\substack{\uparrow \\ \text{motivic extension algebra}}}{E} = \bigoplus_{n \in \mathbb{Z}} \mathrm{Hom}_{DM_G(S)}\left(\mu_!(\mathbb{Q}), \mu_!(\mathbb{Q})(n)[2n]\right) = \left(CH^\bullet_G(X \times_S X)_\mathbb{Q}, *\right)$$

Thm (E.-Stroppel) Assume

(1) $M(\mu^{-1}(\{s\}))$ is pure Tate

(2) $\mu(X)$ has finitely many G-orbits

Then

$$DM_G^{Spr}(S) \xrightarrow{\sim} K^b(DM_G^{Spr}(X)^{w=0}) = D_{perf}(E\text{-mod}^{\mathbb{Z}})$$

Pf: "Yoga of weights"

Geometry of Springer fibers

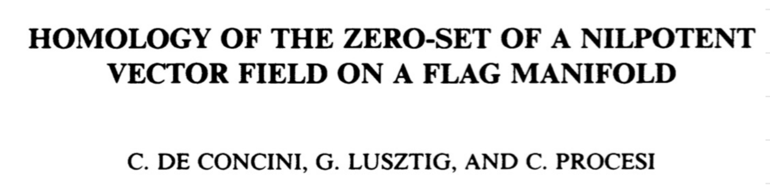
Thm (E. '18) $\mu : X \longrightarrow S$ Springer resolution in all types (A, B, C, D, E, F, G)

$M(\mu^{-1}(A))$ is pure Tate $\forall A \in S$

$\rightsquigarrow$

"Cohomologically, $\mu^{-1}(A)$ looks like a union of affine spaces $\mathbb{C}^n$"

Building on

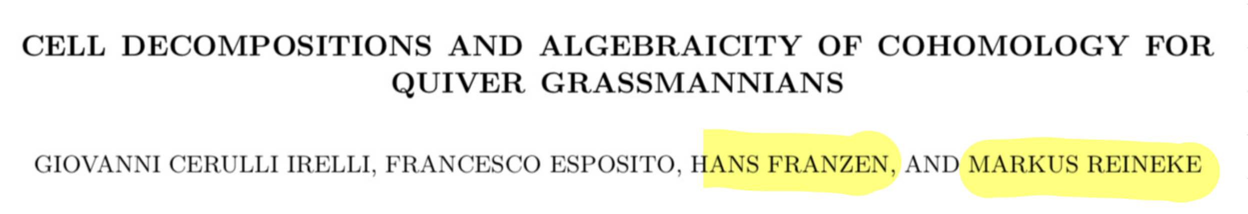


HOMOLOGY OF THE ZERO-SET OF A NILPOTENT VECTOR FIELD ON A FLAG MANIFOLD

C. DE CONCINI, G. LUSZTIG, AND C. PROCESI

Geometry of Partial Quiver Flag Varieties

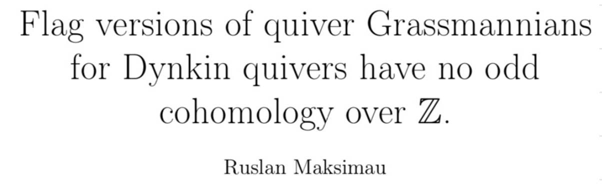
$$\mu : X = \overset{\text{partial}}{\bigsqcup_{\underline{d}}} \overset{\text{quiver flag variety}}{\mathcal{F}l_{\underline{d}}} \longrightarrow S = \mathrm{Rep}(\overset{\text{ADE-quiver}}{Q})$$



CELL DECOMPOSITIONS AND ALGEBRAICITY OF COHOMOLOGY FOR QUIVER GRASSMANNIANS

GIOVANNI CERULLI IRELLI, FRANCESCO ESPOSITO, HANS FRANZEN, AND MARKUS REINEKE

ADE quiver Grassmannian



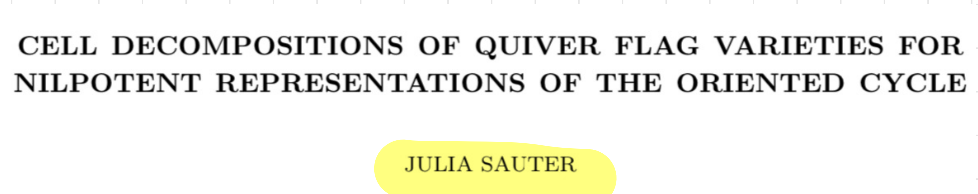
Flag versions of quiver Grassmannians for Dynkin quivers have no odd cohomology over $\mathbb{Z}$.

Ruslan Maksimau

AD partial quiver flag variety

Thm: (Zhou)

ADE partial quiver flag variety



CELL DECOMPOSITIONS OF QUIVER FLAG VARIETIES FOR NILPOTENT REPRESENTATIONS OF THE ORIENTED CYCLE

JULIA SAUTER

$\tilde{A}$ cyclic complete quiver flag variety

Thm (E.-Stroppel)

$\tilde{A}$ cyclic partial quiver flag variety

Applications

Cor. $\mu : T^*G/B \to \mathcal{N}$

$$DM^{Spr}_{G \times G_m}(\mathcal{N}) = D^b(\underset{\uparrow}{\mathbb{H}}(G)\text{-mod}^{\mathbb{Z}})$$

Lusztig's graded affine Hecke algebra

Cor $\mu : \bigsqcup_{\underline{d}} \mathcal{F}l_{\underline{d}} \to Rep(Q)_d$, Q ADE or $\tilde{A}$ quiver

$$DM^{Spr}_{GL(d) \times G_m}(Rep(Q)_d) \cong D^b(R_d\text{-mod}^{\mathbb{Z}})$$

Cor: $\mu : \bigsqcup_{\underline{w} \in W} BS(\underline{w}) \to G/B$, $C = CH^\bullet(G/B)_{\mathbb{Q}}$

$$DM^{Spr}(G/B) \xrightarrow{\sim} K^b(C\text{-SMod}) = D^b(\mathcal{O}(g^L))$$

Further Directions

- Geometric representation theory ⟶ Motivic representation theory
- Generalized motivic cohomology theories

  ⇝ hierarchy of rep. th. based on cohomology theories
- K-theoretic sheaves = K-motives

Conj.

$$\left\{\begin{array}{ll} DK^{Spr}_{G\times \mathbb{G}_m}(\mathcal{N}) & = \text{~~graded~~ affine Hecke algebra} \\ DK^{Spr}_{B}(G/B) & = \text{K-theory Soergel bimodules} \\ DK^{Spr}_{G(\mathcal{O})}\left(G(\mathcal{K})/G(\mathcal{O})\right) & = U_q(\mathfrak{g}^L)\text{-equivariant } \mathcal{O}_q(G^L)\text{-modules} \\ & \vdots \end{array}\right.$$

Thank you 💖