Representation Theory of Linear Algebraic Groups : II

Lecture 1

Content:

0. Algebraic Geometry basics

1. Linear algebraic groups

2. Reductive groups and flag varieties

3. Representation theory

Prerequisits:

- semisimple Lie groups/algebras
- basic commutative algebra

0. Affine and projective varieties

zero set of polynomials/affine variety

coordinate ring

maximal ideals

non-reduced rings

quasi-projective varieties

prime ideals

affine schemes

schemes

functor of points

stacks

⋮

For simplicity we will assume that $k = \overline{k}$ is an algebraically closed field and later that $\operatorname{char} k = 0$.

0.1 Zariski topology and affine varieties

Defn: Let $I \subset k[x_1, \dots, x_n]$ an ideal. The vanishing set of I is defined by

$$V(I) = \{ x \in k^n \mid f(x) = 0 \ \forall f \in I \}.$$

We can equip k^n with the Zariski topology, where the closed sets are exactly sets of the form $V(I)$.

Remark: To see that this is a topology use that

$$\bigcap_\alpha V(I_\alpha) = V\left(\sum I_\alpha\right)$$

$$V(I) \cup V(J) = V(I \cap J)$$

Example: (1) Let $n=1$, then $I \subset k[X]$ is of the form $I = (f)$ and $f = \prod_{i=1}^{n} (x - \lambda_i)$. Then $V(I) = \{\lambda_1, \dots, \lambda_n\}$. Hence Zariski closed sets are just finite sets.

If k is infinite, then k with the Zariski-topology is hence not Hausdorff.

(2) Consider $I = (x), I' = (x^2)$ in $k[x]$.

Then $V(I) = V(I')$ but $I \neq I'$.

(3) Recall that for $I \subset R$ an ideal in a ring the <u>radical</u> of I is defined as

$$\sqrt{I} = \{ x \in R \mid x^r \in R \text{ for some } r \geq 0 \}.$$

Moreover I is <u>radical</u> if $I = \sqrt{I}$.

Note that $V(I) = V(\sqrt{I})$.

<u>Thm</u> (Hilberts Nullstellensatz) There are inclusion-reversing bijections

$$\begin{tikzcd} \{\text{radical ideals } I = \sqrt{I} \subset k[x_1, \dots, x_n]\} \arrow[d, shift left, "V"] \\ \{\text{closed subsets in } k^n\} \arrow[u, shift left, "I"] \end{tikzcd}$$

where $I(X) = \{f \mid f(x) = 0 \ \forall x \in X\}$.

<u>Remark B</u>: Properties of closed subset $X \subset k^n$ as a topological space (equipped with the subspace topology) correspond to properties of the ideal $I = I(X)$

(1) We have

$$X \text{ irreducible} \Leftrightarrow I \text{ is a prime ideal.}$$

Moreover a decomposition into irreducible components

$$X = X_1 \cup \cdots \cup X_n$$

corresponds to the primary decomposition

$$I = I_1 \cap \cdots \cap I_n \qquad \square$$

(2) Points in $a = (a_1, \dots, a_n) \in k^n$ correspond to maximal ideals $m_a = (x - a_1, \dots, x - a_n)$. Similarly,

$$a \in X \iff m_a \supset I(X) \quad .$$

<u>Example B</u>: (1) Let $I = (xy) \subset k[x,y]$. Then $X = V(I)$ union of $X_1 = V(x)$ and $X_2 = V(y)$.

(2) Let $X = V(\, f = x^2 - y) = \{(a,b) \mid a^2 - b\}$

Let $m_{(a,b)} = (x - a, y - b)$. We can expand f at (a,b) and see that

$$x^2 - y = ((x-a)+a)^2 - ((y-a)+a) = a^2 - b$$

$$+ \text{ terms divisible in } m_{(a,b)} \in f(a,b) + m_{(a,b)}$$

Hence $(a,b) \in X \iff f(a,b) = 0 \iff \mathfrak{m}_{(a,b)} \ni f$. $\square$

<u>Def B</u> We call a closed set $X \subset k^n$ an <u>affine variety</u>. D

Warning: many authors define affine varieties as <u>irreducible</u> subsets

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0.2 Regular functions and coordinate rings

We want to define "polynomial" functions between affine varieties. A <u>regular function</u> $F: k^n \to k^m$ is simply a tuple of of polynomials

$$F = (f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n)) \in k[x_1, \ldots, x_n]^m.$$

Similar a regular function between two closed subsets $X \subset k^n$, $Y \subset k^m$ is given by a function F that lifts to a regular function on the ambient spaces

$$\begin{array}{ccc} k^n & \xrightarrow{\tilde{F}} & k^m \\ \cup & & \cup \\ X & \xrightarrow{F} & Y \end{array}$$

We may hence define a category AffVar_k with objects affine varieties X and morphisms

$\mathrm{Hom}(X,Y) = \{F: X \to Y \mid F \text{ is regular}\}$

Def: Let $X \subset k^n$ be a closed subset. Then we define

$$\mathcal{O}(X) = \mathrm{Hom}(X, k)$$

$$= \{ f : X \to k \mid \exists \tilde{f} \in k[x_1, \dots, x_n] \text{ with } \tilde{f}(x) = f(x) \ \forall x \in X \}$$

the <u>coordinate ring</u> of X as the ring of regular functions of $X \to k$. $\square$

Prop: (1) $\mathcal{O}(k^n) = k[x_1, \dots, x_n]$

(2) The natural restriction map

$$k[x_1, \dots, x_n] \to \mathcal{O}(X), \quad f \mapsto f|_X$$

yields an isomorphism

$$k[x_1, \dots, x_n]/I(X) \to \mathcal{O}(X)$$

Proof: (1) Clear. (2) Important exercise! $\square$

If $X \subset k^n$ is an affine variety

$$\mathcal{O}(X) \cong k[x_1, \dots, x_n]/I(X)$$

is a finitely generated reduced k-algebra

We denote by $k\text{-alg}_{red}$ the category of such k-algebras (where morphisms are morphisms of k-algebra)

We obtain the following foundational theorem.

Thm: The assignment $X \mapsto \mathcal{O}(X)$ yields an equivalence of categories

$$\mathrm{AffVar}_k^{op} \longrightarrow k\text{-alg}_{red}.$$

which is defined on morphisms via

$$\mathrm{Hom}(X, Y) \longrightarrow \mathrm{Hom}(\mathcal{O}(Y), \mathcal{O}(X))$$

$$F \longmapsto F^*: h \in \mathcal{O}(Y) \mapsto h \circ F \in \mathcal{O}(X)$$

<u>Proof</u>: Let $A \in k$-alg$_{red}$. By choosing a finite set of generators $x_1, \dots, x_n$, one obtains that $A \cong k[x_1, \dots, x_n]/I$. Then by 0.1. Thm $\mathcal{O}(V(I)) \cong A$.

Hence (up to iso.) A arises from a k-variety.

We need to show that the assignment $F \mapsto F^*$ is a bijection.

We leave this as an extremely important exercise! $\square$

Example: Consider $pt = k^0 = \{0 = pt\}$

$\mathcal{O}(pt) = k$. We obtain bijections

$$\begin{tikzcd}
x \arrow[r, mapsto] & x^* : f \mapsto f(x) \\
X \arrow[r] \arrow[dr] & \mathrm{Hom}(\mathcal{O}(X), \mathcal{O}(pt)) \arrow[d, shift right] \\
x \arrow[dr, mapsto] & \mathrm{Hom}(pt, X) \arrow[u, shift right] \\
& (pt \mapsto x)
\end{tikzcd}$$

0.3. Regular functions on open subsets

Def A: Let X be an affine variety and $f \in \mathcal{O}(X)$.

Then $D(f) = X \setminus V(f)$ is called a principal open set. $\square$

Lemma: The set $D(f) \subset \mathcal{O}(X)$ is an affine variety

with $\mathcal{O}(D(f)) = \mathcal{O}(X)[\frac{1}{f}]$ the localization of

$\mathcal{O}(X)$ at the multiplicative subset $\{1, f, f^2, \ldots\}$

Proof: Let $I(X) = (f_1, \ldots, f_m) \subset k[x_1, \ldots, x_n]$

Define $I' = (f_1, \ldots, f_m, x_{n+1} f - 1)$. Then

$D(f)$ can be identified with $V(I') \subset k^{n+1}$ and

$$\mathcal{O}(D(f)) = k[x_1, \ldots, x_{n+1}] / (f_1, \ldots, f_m, x_{n+1} f - 1)$$
$$\xrightarrow{\sim} \mathcal{O}(X)[\tfrac{1}{f}] \text{ where } x_{n+1} \mapsto \tfrac{1}{f}$$

$\square$

Remk The sets $D(f)$ form a basis of the Zariski topology.

This means that each open $U \subset X$ is of the form

$\bigcup_i D(f_i)$. By Hilberts Basissatz one might even take

finite unions $\square$

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Def B: Let $U \subset X$ be an open subset. We call

a function $F : U \to k$ regular if $F|_{D(f)}$ is

regular for all $D(f) \subset U$. We define

$$\mathcal{O}(U) = \mathcal{O}_X(U) = \{ f : U \to k \mid f \text{ is regular} \} \quad \square$$

Rem B: Regular functions on U can also be defined via

$$\mathcal{O}(U) = \{ f : U \to k \mid \forall x \in X \ \exists x \in V \subset U \text{ open}$$

$$\text{and } p, q \in k[x_1, \ldots, x_n] \text{ s.t. } f(y) = \frac{p(y)}{q(y)} \ \forall y \in V \}.$$

So regular functions on open ~~sets~~ can locally be written as rational functions (quotients of polynomials)

Note that $f(y) = \frac{p(y)}{q(y)}$ implies that $q(y) \neq 0$ for all $y \in V$, so we may only divide by polynomials that don't vanish on V $\square$

0.4 Sheaves of functions

<u>Def A</u>: Let X be a topological space. A <u>sheaf $\mathcal{F}$ of k-valued functions</u> on X is an assignment

$$U \mapsto \mathcal{F}(U) \subset \{f : U \to k\}$$

of a subset of k-valued functions to each open subset of X, such that,

- $\mathcal{F}(U)$ is a k-vector space $\forall U$
- For $f \in \mathcal{F}(U)$ and $V \subset U$, $f|_V \in \mathcal{F}(V)$
- For $f : U \to k$ and $U = \bigcup U_i$ such that $f|_{U_i} \in \mathcal{F}(U_i)$ $\forall i$, $f \in \mathcal{O}(U)$

If additionally $\mathcal{F}(U)$ is closed under multiplication, we call $\mathcal{F}$ a sheaf of rings and $(X, \mathcal{F})$ a (concrete) <u>ringed space</u> (of k-valued functions).

A morphism of ringed spaces

$$F : (X, \mathcal{F}) \to (Y, \mathcal{G})$$

is a continuous map $F: X \to Y$, such that

$F^* f = f \circ F \in \mathcal{F}(f^{-1}(U))$ for all $f \in \mathcal{G}(U)$ $\square$

Example: (1) Let $k=\mathbb{R}$ and X a topological space, then $\mathcal{C}_X(U)=\{f: U\to\mathbb{R} \mid f \text{ is continuous}\}$ is a sheaf of rings.

(2) Let X be an affine variety, then $\mathcal{O}_X$ is a sheaf of rings called the sheaf of regular functions.

(3) If $(X,\mathcal{F})$ is a sheaf of rings and $Y\subset X$ equipped with the subspace topology, then $(Y,\mathcal{F}|_Y)$ is a sheaf of rings, where

$\mathcal{F}|_Y(U)=\{f: U\to k \mid \exists\, V\subset X,\ \tilde{f}\in\mathcal{F}(V) \text{ such that } f=\tilde{f}|_U\}$.

(4) If X is a topological space with open covers $\{U_i\}$ and $\mathcal{F}_i$ a sheaf of functions for each U_i such that $\mathcal{F}_i|_{U_i \cap U_j} = \mathcal{F}_j|_{U_i \cap U_j}$, then there is a unique concrete sheaf of functions $\mathcal{F}$ on X such that $\mathcal{F}|_{U_i} = \mathcal{F}_i$. Namely, define

$$\mathcal{F}(U) = \{ f : U \to k \mid f|_{U_i \cap U} \in \mathcal{F}(U_i \cap U) \text{ for all } i \}$$

Lem: Let X, Y be affine varieties, then

$F: X \to Y$ is regular if and only if

$F: (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a map of ringed spaces

Proof: Important exercise! $\square$

From now on we identify an affine variety X with $(X, \mathcal{O}_X)$.

The assignment $X \mapsto (X, \mathcal{O}_X)$ is called

localization. In particular, if $(X, \mathcal{O}_X)$ is

an affine variety, all information is already encoded

in the ring $\mathcal{O}_X(X)$ (see 0.2 Thm.)

0.5 Varieties

Def: A ringed space $(X, \mathcal{O}_X)$ is called a <u>prevariety</u> if X has a finite open cover $\{U_i\}$ such that $(U_i, \mathcal{O}_X|_{U_i})$ is an affine variety for each i. A prevariety $(X, \mathcal{O}_X)$ is a <u>variety</u> if for each x_1, x_2 there is an open affine $U \subset X$ containing x_1, x_2. A morphism of (pre-)varieties is a morphism of ringed spaces. We denote the category of varieties (over k) by $\mathrm{Var} = \mathrm{Var}_k$ $\square$

By abuse of notation, we often write $X = (X, \mathcal{O}_X)$

<u>Example</u>: (1) Let $(X, \mathcal{O}_X)$ be a variety and $Y \subset X$ be locally closed, that is, an intersection of a closed and open set. Then $(Y, \underbrace{\mathcal{O}_{X|Y}}_{= \mathcal{O}_Y})$ is a variety (see 0.4 Example (3) for the definition of $\mathcal{O}_{X|Y}$)

(2) Varieties can be glued. If X is a topological space with finite open cover $\{U_i\}$, and $\mathcal{O}_{U_i}$ a sheaf of rings on each U_i, such that $(U_i, \mathcal{O}_{U_i})$ is an affine variety and

$\mathcal{O}_{U_i}|_{U_j} = \mathcal{O}_{U_j}|_{U_i}$ $\forall i,j$. Then we can construct $\mathcal{O}_X$ as in 0.4 Example (4). Then $(X, \mathcal{O}_X)$ a prevariety ✓

(2) The variety $\mathbb{A}^n = \mathbb{A}^n_k = (k^n, \mathcal{O}_{k^n})$ is called the <u>affine n-space</u> (over k)

(3) $\mathbb{A}^1 \setminus \{0\} = V(x) \subset \mathbb{A}^1$ is an affine variety, with $\mathcal{O}(\mathbb{A}^1 \setminus \{0\}) = k[x, x^{-1}]$ the ring of Laurent polynomials

(4) $\mathbb{A}^2 \setminus \{0\}$ is not an affine variety since $\mathcal{O}(\mathbb{A}^2 \setminus \{0\}) = k[x, y]$ (Exercise: Show this) □

Lecture 4

0.6 Projective varieties

The projective n-space (over k) is defined as

$$\mathbb{P}^n = \mathbb{P}^n_k = \{ \underset{\substack{\shortparallel \\ \text{1-dim sub vector space}}}{\text{lines}} \text{ in } k^{n+1} \}.$$

We denote by $[a_0 : a_1 : \cdots : a_n] \in \mathbb{P}^n$ the line spanned by a vector $0 \neq (a_1, \dots, a_n) \in k^{n+1}$. In particular, $[a_0 : a_1 : \cdots : a_n] = [b_1 : \cdots : b_n]$ iff there is a $0 \neq \lambda \in k$, s.t., $a_i = \lambda b_i$.

For each i, there is an injective map

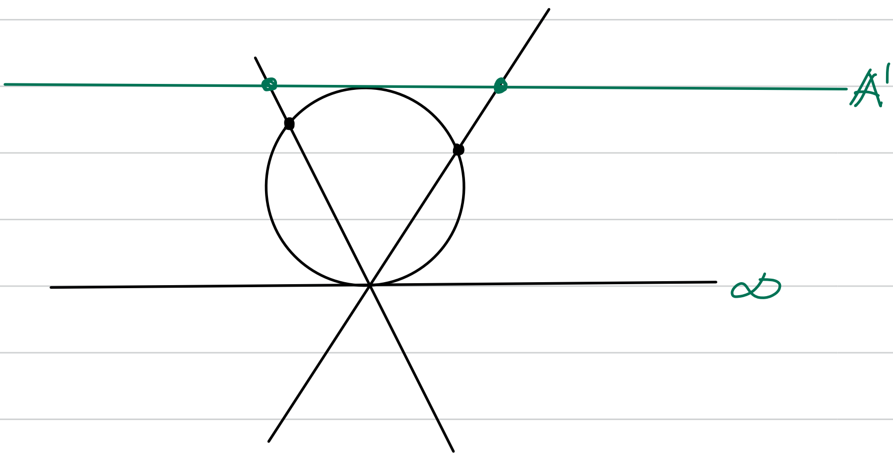
$$\varphi_i : \mathbb{A}^n \longrightarrow \mathbb{P}^n \quad (x_0, \dots, x_{n-1}) \mapsto [x_0 : \cdots : x_{i-1} : 1 : x_i : \cdots : x_{n-1}]$$

The image of this set is $U_i = \{[x_0 : \cdots : x_n] \mid x_i \neq 0\}$.

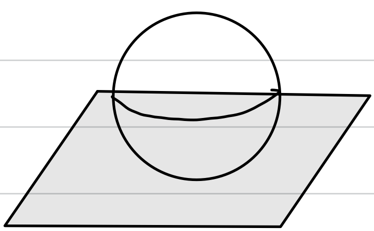
Example: $\mathbb{P}^1$ can be interpreted as $\mathbb{A}^1 \cup \{\infty\}$ by

$\mathbb{P}^1 \to \mathbb{A}^1 \cup \{\infty\}$ $[x:y] \mapsto \frac{x}{y}$, where we

define $\frac{x}{y} = \infty$ for $y=0$. Over $\mathbb{R}$, we may draw



Similarly over $\mathbb{C}$



We can define a topology on $\mathbb{P}^n$ by gluing the Zariski topologies on each U_i, and even obtain a ringed space $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n})$ by gluing the sheaves $\mathcal{O}_{U_i}$, see 0.5 Example (2).

Hence $\mathbb{P}^n = (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n})$ is a variety.

There is also a more direct definition.

Recall that $F \in k[x_0, \dots, x_n]$ is called homogeneous, if there is a d (the degree), s.t. all monomials in F have degree d.

In particular $F(\lambda x_0, \dots, \lambda x_n) = \lambda^d F(x_0, \dots, x_n)$.

An ideal $I \subset k[x_0, \dots, x_n]$ is called homogeneous if I is generated by homogeneous polynomials.

We define

$$V_{\mathbb{P}^n}(I) = \{ [x_0 : \dots : x_n] \mid \forall f \in I \text{ homogeneous} \}.$$

These form the closed sets of the Zariski topology on $\mathbb{P}^n$. Moreover

$\mathcal{O}_{\mathbb{P}^n}(U) = \{ f : \mathbb{P}^n \to k \mid \forall x \in U \; \exists x \in V \subset U$ and $p, q \in$

$k[x_0, \dots, x_n]$ homogeneous of degree d, such that

$$f(y) = \frac{p(y)}{q(y)} \quad \forall y \in V \}$$

Example: (1) Let $f = \sum_{\lambda} a_\lambda x_1^{\lambda_1} \cdots x_n^{\lambda_n} \in k[x_1, \ldots, x_n]$.

Then we can homogenise f via

$$F = \sum_{\lambda} a_\lambda x_0^{d - \sum \lambda_i} \quad x_1^{\lambda_1} \cdots x_n^{\lambda_n}$$

and $d = \max_{\lambda} \sum \lambda_i$. Then $f = F(1, x_1, \ldots, x_n)$.

Then $V_{\mathbb{P}^n}(F) \cap U_0 = V(f) \subset \mathbb{P}^n$.

(2) Let $f = x^2 - y$. Then $F = x^2 - wy$, and

$$V(F) = \{[c : a : b] \mid a^2 - cb\} \qquad a^2 = 0$$

$$= \underbrace{\{[1 : a : a^2]\}}_{\substack{\cong \\ \mathbb{A}^1}} \cup \underbrace{\{[0 : 0 : 1]\}}_{\substack{\| \\ pt}}$$

$\cong \mathbb{P}^1$

Study conic sections in this language!

(3) Projective n-space admits a decomposition into locally closed subspaces

$$\begin{array}{ccccc} & & \{[1:a_1:\cdots:a_n]\} & & \{[0:a_1:\cdots a_n]\} \\ & & \| & & \| \\ \mathbb{P}^n & = & \mathbb{A}^n & \sqcup & \mathbb{P}^{n-1} \end{array}$$

$$= \mathbb{A}^n \sqcup \mathbb{A}^{n-1} \sqcup \cdots \sqcup \mathbb{A}^1 \sqcup \mathbb{A}^0$$

(4) Recall that $U_i = \{[x_0 : \cdots x_n] \mid x_i \neq 0\}$. Then

$$\mathcal{O}(U_i) = \left\{ \frac{f}{x_i^d} \;\middle|\; f \in k[x_0, \ldots, x_n], \text{ hom. of deg. } d \right\}$$

$$= k\left[\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}\right] \cong k[y_1, \ldots, y_n]$$

<u>Exercise</u>: Use this and the fact that $\mathcal{O}_{\mathbb{P}^n}$ is a sheaf to show that $\mathcal{O}_{\mathbb{P}^n}(\mathbb{P}^n) = k$ $\square$

Def: Let $X=(X, \mathcal{O}_X)$ be a variety. Then

(1) X is called projective if X is a closed subset of $\mathbb{P}^n$

(2) X is called quasiprojective (or quasi-affine) if it is a locally closed subset of $\mathbb{P}^n$ (or $\mathbb{A}^n$) $\square$

Rem: The spaces $\mathbb{A}^n$ and $\mathbb{P}^n$ can also be defined "coordinate-free" for each vector space V.

So we can define $(V, \mathcal{O}_V)$ where $\mathcal{O}_V(V) = \mathrm{Sym}(V^*)$. Similar, we write $\mathbb{P}_V = (\mathbb{P}_V, \mathcal{O}_V)$. $\square$

Lecture 5

0.8. Grassmannians as projective varieties

<u>Idea</u>: Express subspace relations in a vector space V via $\mathbb{P}(\Lambda^\bullet V)$. $\square$

<u>Def</u>: Let V be an n-dim. vector space. The (set-theoretical) Grassmannian of d-dim. subspaces in V is

$$Gr_d(V) = \{ W \subset V \mid \dim W = d \}$$ $\square$

Prop: Let V be a n-dim vector space.

Then, the map, called Plücker embedding,

$$i: \quad Gr_d(V) \to \mathbb{P}(\Lambda^d V)$$

$$W = \langle w_1, \dots, w_d \rangle \longmapsto [\omega = w_1 \wedge \dots \wedge w_d]$$

is well-defined and injective, where

$W = \{ v \in V \mid \omega \wedge v = 0 \}$.

Proof: Let $g \in GL(W)$, then

$$g w_1 \wedge \dots \wedge g w_d = (\det g)\, w_1 \wedge \dots \wedge w_d$$

and hence both define the same element in $\mathbb{P}(\Lambda^k(V))$.

Injectivity and the description of W is an exercise

Hint: Complete a basis of W to V $\square$

Call a vector $0 \neq v \in \Lambda^d V$ simple if there are $v_i \in V$, s.t.,

$v = v_1 \wedge \dots \wedge v_d$. In particular,

$$c(Gr_d(V)) = \{[\omega] \mid \omega \in \Lambda^d V \text{ simple}\}.$$

The following yields an algebraic description of simplicity.

Thm: (Plücker relations) Let $0 \neq v \in \Lambda^d V$. Then

$$v \text{ simple} \iff v \wedge i_\alpha(v) = 0 \quad \forall \alpha \in \Lambda^{d-1} V^*$$

Proof: Exercise (involved) $\square$

Exercise: (Involved)

Choosing a basis of V, write the condition

$$v \wedge W = 0$$

as a set of homogeneous quadratic relations in $\Lambda^d V$.

Then coordinatize $\Lambda^d V$ in terms of $d \times d$-minors.

Combined, obtain the classical Plücker relations.

Example: Let $V = k^4 = \langle e_1, \dots, e_4 \rangle_k$ and $d = 2$.

Let $W = \langle v, w \rangle \in Gr_2(V)$. Put v, w in a matrix

$$\begin{pmatrix} a_1 & \cdots & a_4 \\ b_1 & \cdots & b_4 \end{pmatrix}$$

and denote the 2×2-minors by

$$p_{ij} = a_i b_j - a_j b_i$$

Then one obtains that

$$\omega = v \wedge w = \sum_{i<j} p_{ij}\, e_i \wedge e_j$$

Now $\Lambda^2 V$ has basis $\{e_i \wedge e_j \mid i<j\}$, so we may identify $\mathbb{P}(\Lambda^2 V) = \mathbb{P}^{\binom{4}{2}-1} = \mathbb{P}^5$ and we call

$$[p_{12} : \dots : p_{34}] \in \mathbb{P}^5$$

the Plücker coordinates. Let us now compute the relations amongst the Plücker coordinates explicitly.

By Thm, these are exactly of the form

$$i_\alpha(\omega) \wedge \omega = 0 \qquad \text{for all } \alpha \in \Lambda^{d-1} V^* = V^*.$$

Let $\alpha = e_1^*$. By definition

$$i_\alpha(e_i \wedge e_j) = \alpha(e_i)\, e_j - \alpha(e_j)\, e_i$$

so that

$$i_{e_1^*}(\omega) = p_{12} e_2 + p_{13} e_3 + p_{14} e_4$$

Moreover, one may expand

$$\omega \wedge i_{e_1^*}(\omega) = \underbrace{(p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23})}_{R}\, e_2 \wedge e_3 \wedge e_4 .$$

So $R = 0$. Moreover, $\alpha = e_2, e_3, e_4$ yields a similar expression involving $\pm R$. So R is the only Plücker relation, so that $Gr_d(V) \subset \mathbb{P}^5$ is a hypersurface given by $R=0$, also known as the Klein Quadric $\square$

<u>Remark</u>: Along these lines one may also show that for any sequence $\underline{d} = 0 < d_1 < \cdots < d_k < d$ the <u>flag variety</u> $\mathcal{F}\ell_{\underline{d}}(V) = \{ V_1 \subset \cdots \subset V_k \subset V \mid \dim V_i = d_i \}$ is projective $\square$

0.9 Function field, local rings and dimension

Assume that $X=(X,\mathcal{O}_X)$ is irreducible.

<u>Def A</u>: The <u>function field</u> of X is defined as

$$k(X) = \mathcal{O}_{X,\eta} = \operatorname*{colim}_{U\neq\emptyset} \mathcal{O}_X(U)$$

$$= \{(U,f) \mid \emptyset \neq U \subset X,\ f: U \to k \text{ regular}\}/\sim$$

Let $p \in X$. Then the <u>local ring</u> at p is

$$\mathcal{O}_{X,p} = \operatorname*{colim}_{p\in X} \mathcal{O}_X(U)$$

$$= \{(U,f) \mid \emptyset \neq U \subset X,\ f: U \to k \text{ regular}\}/\sim$$

Here, $(U,f)\sim(U',f')$ if $f|_W = f'|_W$ for some $W \subset U \cap U'$.

Remark A: If $\emptyset \neq U \subset X$ is open, then $k(X) = k(U)$, similarly $\mathcal{O}_{X,p} = \mathcal{O}_{U,p}$. We may hence assume that X is affine. In this case, we have

$$k(X) = \operatorname{Frac}(\mathcal{O}(X)) \quad \text{and}$$

$$\mathcal{O}_{X,p} = \{ \tfrac{f}{g} \in k(X) \mid g(p) \neq 0 \}$$

$$= \mathcal{O}(X)_{m_p}$$

Moreover, $\mathcal{O}_{X,p}$ is a local ring with maximal ideal m_p. One often considers the completion

$$\hat{\mathcal{O}}_{X,p} = \lim_n \mathcal{O}_{X,p}/m_p^n$$

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Example: Let $X = \mathbb{A}^n$ or $X = \mathbb{P}^n$ and $p = 0$.

$$k(X) = k(x_1, \dots, x_n)$$

is the ring of rational functions in n variables

$$\mathcal{O}_{X,p} = k[x_1, \dots, x_n]_{(x_1, \dots, x_n)}$$

and the completion is the ring of power series

$$\hat{\mathcal{O}}_{X,p} = k[[x_1, \dots, x_n]]$$

(2) Let $X = V(y^2 - x^3 + x)$ an elliptic curve

(assume char $k \neq 2, 3$). Then

$$k(X) = k(x)(\sqrt{x^3 - x})$$

is a quadratic extension of $k(x)$.

Moreover

$\mathcal{O}_{X,p} = \{ \frac{f}{g} \mid g(p) \neq 0 \}$ and $\hat{\mathcal{O}}_{X,p} = k[\![t]\!]$, $t = x - p$

So, locally X behaves very similarly to $\mathbb{A}^1$!

(3) Let $X = V(y^2 - x^3 - x^2)$, a <u>nodal cubic</u>

then one may check that

$$k(X) = k[t] \quad \text{for } t = \frac{x}{y}$$

so has the same function field as $\mathbb{A}^1$. We

say that X and $\mathbb{A}^1$ are <u>birational</u> ($\Leftrightarrow$ same

function field $\Leftrightarrow$ isomorphic on a dense open subset).

Also X is <u>rational</u> ($\Leftrightarrow$ birational to $\mathbb{A}^n$

for some n).

Exercise: Which examples of X rational have we seen so far?

Something special happens at $p=(0,0)\in X$.

One may show that

$$\mathcal{O}_{X,p}=k[[x,y]]/((y-x\sqrt{x+1})(y+x\sqrt{x+1}))$$

$$\cong k[[u]]\times k[[v]]$$

so locally at p, X look like the coordinate cross $V(uv)\subset\mathbb{A}^2$. In particular X is not smooth at $p=(0,0)$. We will discuss this more in the next section.

Def B: Let X be a variety then the dimension of X is given by

$$\dim X = \sup\{ n \mid \exists \text{ chain } X \supsetneq X_1 \supsetneq X_2 \supsetneq \cdots \supsetneq X_n \neq \emptyset \text{ with}$$
$$X_i \text{ irreducible and closed} \}$$

so if X is affine and irreducible

$$\dim X = \dim \mathcal{O}(X)$$

is the Krull-dimension of the ring $\mathcal{O}(X)$

For a point $p \in X$, the dimension of X at p is defined as

$$\dim_p X = \dim \mathcal{O}_{X,p}$$

$\square$

<u>Remark B</u>: If X is irreducible, then for all $p \in X$

$$\dim X = \dim \mathcal{O}_{X,p} = \dim \hat{\mathcal{O}}_{X,p}.$$

Moreover $\dim \mathbb{A}^n = n$.

<u>Exercise</u> Use Remark B to compute the dimension of the varieties in the last example.

0.10 Tangent space

Again let X be an irreducible variety.

<u>Def A:</u> Let $p \in X$. Then the <u>tangent space</u> of X at p is defined as

$$T_pX = (m_p/m_p^2)^*$$

where $(\ )^*$ is the k-linear dual and $m_p \subset \mathcal{O}_{X,p}$ the maximal ideal. $\square$

The definition is local in $p \in X$. Hence, by passing to a local affine neighborhood of p, we may <u>assume</u> that $X = V(I) \subset \mathbb{A}^n$ is <u>affine</u>.

Remark: (1) Let $p \in X$. Then the maps

$$\mathcal{O}(X) \to \mathcal{O}_{X,p} \to \hat{\mathcal{O}}_{X,p}$$

induce an isomorphism between the respective quotients m_p/m_p^2.

(2) Intuition: For $p \in X$, a regular function can be expanded as

$$f = f(p) + \underbrace{\text{linear terms}}_{\in m_p/m_p^2} + \underbrace{\text{higher order terms}}_{\in m_p^2}$$

Hence $\lambda \in T_pX$ corresponds to a directional derivative, see the next point.

(3) The tangent space T_pX is isomorphic to the space of k-linear maps (point derivations)

$$\delta : \mathcal{O}(X) \to k,$$

fulfilling the Leibniz rule

$$(*) \quad \delta(fg) = \delta(f)\, g(p) + f(p)\, \delta(g),$$

where $\lambda \in (m_p/m_p^2)^* \Leftrightarrow \delta_\lambda = \lambda(f - f(p))$.

(4) Recall that we identified in 0.2 Example

$$X \cong \mathrm{Hom}(pt, X) \cong \mathrm{Hom}(\mathcal{O}(X), k).$$

So, more generally, for a k-algebra A, we set

$$X(A) = \mathrm{Hom}(\mathcal{O}(X), A)$$

so that $X(k) = X$. Assume that

$$I = (f_1, \dots, f_m) \subset k[x_1, \dots, x_n]$$

then $X(A) = \{(a_1, \dots, a_n) \in A^n \mid f_i(a_1, \dots, a_m) = 0 \ \forall i\}$.

A map of algebras $A \to B$ yields a map

$X(A) \to X(B)$. Now consider

$k[\varepsilon]/\varepsilon^2$ the dual numbers, then

$\varphi \in X(k[\varepsilon]/\varepsilon^2) = \mathrm{Hom}(\mathcal{O}(X), k[\varepsilon]/\varepsilon^2)$

is of the form

$$\varphi(f) = f(p) + \delta(f)\varepsilon.$$

where $p \in X = X(k)$ and the requirement that

φ is a ring homomorphism is exactly the Leibniz rule $(*)$. The map $k[\varepsilon]/\varepsilon^2 \to k$ yields a map $\pi: X(D) \to X(k)$ and $T_pX = \pi^{-1}(p)$

Lecture 7

(5) For $X = V(f_1, \dots, f_m) \subset \mathbb{A}^n$ and $p \in X$, the <u>Jacobian matrix</u> at p is defined as

$$J_p = \left(\frac{\partial f_i}{\partial x_j}(p)\right) \in k^{m \times n}$$

and $T_pX \cong \ker J_p \subset k^n$.

(6) The tangent space is functorial in the following sense

If $f: X \to Y$ is a morphism of varieties, then the differential of f at p is a k-linear map

$$df_p : T_pX \to T_pY.$$

<u>Exercise</u>: Guess the definition of df_p in all previous descriptions of T_pX.

(7) One may show that

$$\dim_k T_pX \geq \dim_p X$$

with equality on a dense open subset. $\square$

Def B: A point $p \in X$ is smooth if

$$\dim_k T_p X = \dim_p X$$

X is called smooth if all points in X are smooth.

Example A (1) let $X = \mathbb{A}^n$ and $p = (0, .., 0) \in \mathbb{A}^n$

$$m_{X,p} = (x_1, .., x_n) \text{ and } m_{X,p}/m_{X,p}^2 = \bigoplus_{i=1}^n k\, dx_i$$

where $dx_i = \overline{x}_i \in m_{X,p}/m_{X,p}^2$ (similar for all $p \in \mathbb{A}^n$).

So $\dim T_p X = n$, so $\mathbb{A}^n$ is smooth.

(2) Now let $X = V(f_1, .., f_m) \subset \mathbb{A}^n$. Assume that $p = (0, .., 0) \in X$. (Else shift the coordinates). Denote by $m = (x_1, .., x_n)$ the maximal ideal of p in $\mathbb{A}^n$.

Then one may identify

$$\mathfrak{m}_{X,p}/\mathfrak{m}_{X,p}^2 \cong \mathfrak{n}/(\mathfrak{n}^2+(f_1,\dots,f_m))$$

$$\cong \mathfrak{n}/(\mathfrak{n}^2+(d_0f_1,\dots,d_0f_i))$$

$$= \bigoplus_{i=1}^{n} k\,dx_i \Big/ \bigoplus_{i=1}^{m} k\,d_0f_i$$

where $d_0f = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(0)\, dx_i$ is the linear part of f in the Taylor series of f at $p=0$

Exercise: Show that $T_pX \cong \ker J_p$ with this idea.

(3) Let $X = V(f) \subset \mathbb{A}^2$ where $f = y^2 - x^3 - x$ and $p = (0,1)$. Write $u = x$, $v = y-1$. Then

$$f = 2v + v^2 - u - u^3 = \underbrace{(-u+2v)}_{\text{linear!}} + v^2 + (-u^3)$$

So $d_p f = -du + 2dv$

So we can identify

$$m_p/m_p^2 = k\,du \oplus k\,dv / k\,d_p f \cong k.$$

Hence the point $p=(0,1)$ is smooth. Also

$\nabla f = (-3x^2-1, 2y)$ and $J_p = \nabla f(0,1) = (-1,2)$

has full rank.

(4) Let $X = V(f)$ for $f = y^2 - x^3 - x^2$ a nodal cubic.

and $p = (0,0) \in X$. Then

Then $\nabla f = (3x^2-2x, 2y)$, so $J_p = \nabla f(0,0) = (0,0)$.

Hence $\dim T_{X,p} = 2 > \dim X = 1$ and p is singular! $\square$

The following shows that smooth point in the completion, look like points in $\hat{\mathbb{A}}^d$, which is an algebraic version of the implicit function theorem.

Thm (Cohen) If $p \in X$ is smooth and $d = \dim X$, then

$$\hat{\mathcal{O}}_{X,p} \cong k[[t_1, \dots, t_d]]$$ $\square$

0.11 Complements

Varieties are closed under fiber products

$$\begin{tikzcd}
W \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& X \times_Z Y \arrow[r, "p_X"] \arrow[d, "p_Y"] & X \arrow[d, "f"] \\
& Y \arrow[r, "g"] & Z
\end{tikzcd}$$

with underlying set of points

$$X \times_Z Y = \{ (x,y) \mid f(x) = f(y) \}$$

For $Z = pt$, we obtain the Cartesian product

$$X \times_{pt} Y = X \times Y$$

For $g : Y = \{pt\} \to X$, $g(pt) = z$ we obtain the fibers

$$X \times_Z \{pt\} = f^{-1}(z)$$

If X, Y, Z are affine

$$\mathcal{O}(X \times_Z Y) = \left(\mathcal{O}(X) \otimes_{\mathcal{O}(Z)} \mathcal{O}(Y)\right)_{red}$$

We have $\dim(X \times Y) = \dim X + \dim Y$,

$$\mathbb{A}^n \times \mathbb{A}^m \to \mathbb{A}^{n+m}$$

$$\mathbb{P}^n \times \mathbb{P}^m \hookrightarrow \mathbb{P}^{(m+1)(n+1)-1} \qquad ([x_i],[y_j]) \mapsto ([x_i y_j])$$

$$\mathbb{P}^n \times \mathbb{P}^m \neq \mathbb{P}^{n+m}$$

Segre embedding

so products preserve (quasi-) affine/projective.

<u>Thm A</u>: For dominant (= dense image) $f: X \to Y$,

$$\dim F \geq \dim X - \dim Y$$

for each $F \subseteq \varphi^{-1}(x)$ irreducible component. The inequality is a equality on an open dense subset. $\square$

A finite union of locally closed subsets is called <u>constructible</u>

<u>Thm B</u> (Chevalley) Let $f: X \to Y$ be a morphism of varieties and $S \subset X$ constructible. Then so is $f(S)$. Moreover S contains a dense open subset of $\bar{S}$. $\square$

<u>Exercise</u>: Consider $f: \mathbb{A}^2 \to \mathbb{A}^2$ $(x,y) \mapsto (x, xy)$.

Compute image and fibers!

Lecture 8

We will also occasionally use

Thm C (Zariski's main theorem) Let $f: X \to Y$ be birational (isomorphism on open dense subsets), quasi-finite (all fibers $f^{-1}(y)$ are finite sets) and Y smooth (normal suffices). Then f is an open immersion. $\square$

Rem: If X and Y are additionally projective and f is bijective on the points of X and Y, then f is an isomorphism $\square$

1 Linear algebraic groups

1.1 Affine algebraic groups

<u>Def</u> A: An <u>affine algebraic group</u> is an affine variety G with morphisms

- $\mu : G \times G \to G$ (multiplication)
- $\iota : G \to G$ (inversion)
- $\epsilon : pt \to G$ (unit element)

such that the following diagrams commute

$$\begin{tikzcd} G \times G \times G \arrow[r, "\mu \times id"] \arrow[d, "id \times \mu"] & G \times G \arrow[d, "\mu"] \\ G \times G \arrow[r, "\mu"] & G \end{tikzcd} \qquad \begin{tikzcd} G \times pt \arrow[r, "id \times \epsilon"] \arrow[dr] & G \times G \arrow[d, "\mu"] & pt \times G \arrow[l, "\epsilon \times id"'] \arrow[dl] \\ & G & \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, "(id, \iota)"] \arrow[d] & G \times G \arrow[d, "\mu"] & G \arrow[l, "(\iota, id)"'] \arrow[d] \\ pt \arrow[r, "\epsilon"] & G & pt \arrow[l, "\epsilon"'] \end{tikzcd}$$

A morphism of affine algebraic groups is a morphism of varieties compatible with the above structure map.

<u>Def</u> B: A commutative Hopf algebra A (over k) is a k-algebra A with morphisms

- $\Delta : A \to A \otimes A$ (comultiplication)
- $\iota : A \to A$ (co-inverse)
- $\varepsilon : A \to k$ (co-unit)

such that the following diagrams commute

$$
\begin{tikzcd}
A\otimes A\otimes A & A\otimes A \arrow[l, "\Delta\otimes id"'] \\
A\otimes A \arrow[u, "id\otimes\Delta"] & A \arrow[l, "\Delta"] \arrow[u, "\Delta"']
\end{tikzcd}
\qquad
\begin{tikzcd}
A\otimes k & A\otimes A \arrow[l, "id\otimes\varepsilon"'] \arrow[r, "\varepsilon\otimes id"] & k\otimes A \\
& A \arrow[ul] \arrow[u, "\Delta"'] \arrow[ur] &
\end{tikzcd}
\qquad
\begin{tikzcd}
A & A\otimes A \arrow[l] \arrow[r] & A \\
k \arrow[u] & A \arrow[l] \arrow[r] \arrow[u, "\Delta"'] & k \arrow[u]
\end{tikzcd}
$$

A morphism of Hopf algebras is ....

An Hopf algebra is called reduced / finitely generated / ...

if the underlying algebra is $\square$

<u>Def C</u>: A k-group functor G is a functor

$$G : \{k\text{-algebras}\} \longrightarrow \{\text{groups}\}$$

from the category of k-algebras to the category of

(abstract) groups. $\square$

<u>Thm</u>: The following are equivalences of categories

$$\begin{tikzcd}
& \{\text{affine alg. grp's}/k\} \arrow[dl, "(1)"'] \arrow[dr, "(2)"] & \\
\left\{\begin{matrix}\text{reduced finitely generated} \\ \text{commutative Hopf algebras}/k\end{matrix}\right\} \arrow[rr, "(3)"] & & \left\{\begin{matrix}\text{group functors representable by} \\ \text{reduced finitely generated alg.}/k\end{matrix}\right\}
\end{tikzcd}$$

(1) $G \mapsto \mathcal{O}(G)$

(2) $G \mapsto \left(B \mapsto G(B) = \mathrm{Hom}_{k\text{-alg}}(\mathcal{O}(G), B)\right)$

(3) $A \mapsto \left(B \mapsto \mathrm{Hom}_{k\text{-alg}}(A, B)\right)$

<u>Proof</u>: This is an application of 0.2 Thm and the Yoneda Lemma $\square$

<u>Examples</u> (1) $G = \mathbb{A}^1 = \mathbb{G}_a$, the <u>additive group</u>:

- $\mu(x,y) = x+y$, $\iota(x) = -x$, $\epsilon(pt) = 0$
- $A = k[x]$, $\Delta(x) = x \otimes 1 + 1 \otimes x$, $\iota(x) = -x$, $\epsilon(x) = 0$
- $\mathbb{G}_a(R) = (R,+)$

(2) $G = \mathbb{A}^1 - \{0\} = \mathbb{G}_m = GL_1$, the <u>multiplicative group</u>

- $\mu(x,y) = xy$, $\iota(x) = x^{-1}$, $\epsilon(pt) = 1$
- $A = k[x,x^{-1}]$, $\Delta(x) = x \otimes x$, $\iota(x) = x^{-1}$, $\epsilon(x) = 1$
- $\mathbb{G}_m(B) = (R^\times, \cdot)$

(3) $\mathcal{G} = \mathcal{GL}_n = D(\det) \subset k^{n\times n}$, the <u>general linear group</u>

- $\mu(X,Y) = XY$, $i(X) = X^{-1}$, $\epsilon(pt) = I_n$

- $\Delta(x_{ij}) = \left(\sum x_{ik} \otimes x_{kj}\right)$ $c(x_{ij}) = (-1)^{i+j} \det(x_{i'j'})_{\substack{i' \neq j \\ j' \neq i}} \det(x_{i'j'})^{-1}$, $\epsilon(x_{ij}) = \delta_{ij}$

- $\mathcal{GL}_n(R) = \{ g \in R^{n\times n} \mid \det(g) \in R^{\times} \}$

(4) For V a f.d. v.s. $/k$, $\mathcal{GL}(V)$ is the functor

$$R \longmapsto \mathcal{GL}(V \otimes_k R) = \{R\text{-linear automorphisms of } V \otimes_k R\}.$$

(5) If $\mathcal{G}$ and $\mathcal{H}$ are affine algebraic groups, then so is

$\mathcal{G} \times \mathcal{H}$

(6) If G is a finite group. Then G defines an affine

algebraic group with underlying variety $\biguplus_{g \in G} pt$ $\square$

Lecture 9

1.2 Basic properties of affine algebraic groups

<u>Prop A</u> Let G be an affine alg. grp.

(1) There is a unique irreducible component $G^\circ \subset G$ with $e \in G^\circ$.

(2) $G^\circ \subset G$ is a normal subgrp of finite index and the cosets are the irreducible components of G.

<u>Proof</u>: (1) Denote by $X_1, \ldots, X_m \subset G$ the distinct irreducible components containing e. Then the image $X_1 \cdots X_m$ $X_1 \times \cdots \times X_m \to G$ is again an irreducible and contains e, so wlog $X_1 \cdots X_m = X_1$, so $X_i = X_1 = G^\circ$.

(2) By (1), $\mathcal{G}^\circ \mathcal{G}^\circ = \mathcal{G}^\circ$, $(\mathcal{G}^\circ)^{-1} = \mathcal{G}^\circ$ and $x \mathcal{G}^\circ x^{-1} = \mathcal{G}^\circ$, since all these sets are irreducible components containing (1).

The cosets $g\mathcal{G}^\circ$ are hence also irreducible.

As an affine variety, $\mathcal{G}$ can only have finitely many irreducible components. $\square$

We call $\mathcal{G}$ <u>connected</u> if $\mathcal{G} = \mathcal{G}^\circ$.

<u>Lemma</u>: If $U, V \subset \mathcal{G}$ are dense open, then $\mathcal{G} = U \cdot V$.

<u>Proof</u>: Let $x \in \mathcal{G}$. Then $U \cap xV^{-1} \neq \emptyset$ by density. Hence $x \in UV$. $\square$

<u>Prop B</u>: Let $H, H' \subset G$ be subgroups of G (as abstract groups). Then

(1) $\overline{H} \subset G$ is also a subgroup

(2) If H is constructible, then $H = \overline{H}$ is closed in G.

(3) If $H, H' \subset G$ are closed and H' normalizes H, then $H \cdot H' \subset G$ is a closed subgroup

<u>Proof</u>: (1) Use that $x\overline{H} = \overline{xH}$ and $\overline{H}^{-1} = \overline{H^{-1}}$.

(2) If H is constructible, it contains an open dense subset $U \subset \overline{H}$. Since H is a group $\overline{H} = U \cdot U \subset H$.

(3) HH' is constructible as the image of $H \times H' \xrightarrow{m} G$

by 0.11 Thm B and moreover a subgroup.

Hence by (2) $HH' \subset G$ is closed $\square$

<u>Prop C</u>: Let $\varphi : G \to G'$ be a morphism of affine algebraic groups. Then

(a) $\ker\varphi \subset G$ and $\operatorname{im}\varphi \subset G'$ are closed subgroups

(b) $\varphi(G^\circ) = \varphi(G)^\circ$

(c) $\dim G = \dim \ker\varphi + \dim \varphi$

<u>Pf</u>: (a), (b) follow from Prop A + B.

(c) follows from 0.11 Thm A $\square$

Prop D. Let $f_i : X_i \to G$ a family of morphisms from irreducible varieties into an affine alg. grp., such that, $e \in Y_i = f_i(X_i)$. Let $M = \bigcup Y_i$ and

$$\langle M \rangle = \bigcap_{\substack{M \subset G' \subset G \\ \text{closed} \\ \text{subgroup}}} G'$$

Then $\langle M \rangle \subset G$ is a connected closed subgroup and

$$\langle M \rangle = Y_{a_1}^{\varepsilon_1} \cdots\cdots Y_{a_n}^{\varepsilon_n}$$

for some sequence of indices a_i and $\varepsilon_i \in \{\pm 1\}$.

In part., if $Y_i \subset G$ are closed connected subgroups of G and $\langle M \rangle = G$, then G is connected $\triangle$

1.3. Actions of affine alg. grp's

We can now upgrade many definitions from the theory of abstract groups to affine algebraic groups.

For example, we can define a G-variety X as a variety X with a G-action $G \times X \to X$ that is a morphism of varieties fulfilling the usual axioms.

We then obtain that stabilizers $G_x \subset G$, centralizers $C_G(H) \subset G$ and normalizers $N_G(H) \subset G$ are closed subgroups (here $H \subset G$ is closed).

Similarly, the fixed point set $X^G \subset X$ is closed.

For G-orbits we obtain the following

<u>Prop</u>: Let X be a G-variety. Then each G-orbit $G\cdot x \subset X$ is smooth and locally closed. The boundary $\overline{G\cdot x} - G\cdot x$ is a union of orbits of strictly lower dimension. Orbits of minimal dimension are closed

<u>Proof</u> Exercise: Use 0.11 Thm B and that smooth points form an open dense subset $\square$

We may also define (algebraic) representations as morphisms $\mathcal{G} \to GL(V)$, which is equivalent to specifying an action of $\mathcal{G}$ on V via linear maps. We then call V an (algebraic) module of $\mathcal{G}$. For V, W $\mathcal{G}$-modules, $V^* \otimes W$ is again a $\mathcal{G}$-module.

If $\mathcal{G}$ is an affine alg. grp. acting on variety X via $\varphi : \mathcal{G} \times X \to X$. Then $\mathcal{G}$ acts k-linearly on $\mathcal{O}(X)$ via $(gf)(x) = f(g^{-1}x)$, for $g \in \mathcal{G}$, $x \in X$, $f \in \mathcal{O}(X)$.

The action map φ yields

$$\varphi^* : \mathcal{O}(X) \to \mathcal{O}(\mathcal{G} \times X)$$

$$f \mapsto ((g,x) \mapsto f(g \cdot x))$$

Now we identify

$$\mathcal{O}(\mathcal{G} \times X) \cong \mathcal{O}(\mathcal{G}) \otimes \mathcal{O}(X)$$

$$(g,x) \mapsto a(g)b(x) \leftarrow\!\shortmid a \otimes b$$

So if $\varphi^*(f) = \sum_{i=1}^{n} a_i \otimes b_i$, then

$$\varphi^*(f)(g, x) = \sum_{i=1}^{n} a_i(g)\, b_i(x)$$

and $(g \cdot f)(x) = f(g^{-1}x) = (\varphi^*(f))(g^{-1}, x) = \sum a_i(g^{-1})\, b_i(x)$

In this language, a G-variety is the same as a (reduced, f.g.) Hopf-comodule over $\mathcal{O}(G)$.

Lecture 10

The left and right action of G on itself hence induces actions on $\mathcal{O}(G)$ via

$$(\lambda_g f)(x) = f(g^{-1}x)$$

$$(\rho_g f)(xg) = f(xg)$$

which we will study now.

1.4: Linearization

Def: A linear algebraic group is a closed subgroup of GL_n

A linear alg. grp. is affine. We now show the converse.

Lemma: Let G be an aff. alg. group acting on an affine variety X via $\varphi : G \times X \to X$. Then

(1) for each $f \in \mathcal{O}(X)$, there is a f.d. G-stable subspace such that $f \in V \subset \mathcal{O}(X)$

(2) X admits a G-eq. closed immersion $X \hookrightarrow V$ into a f.d. G-rep. V

Proof: (1) Write $\varphi(f) = \sum_{i=1}^{n} a_i \otimes b_i \in \mathcal{O}(G) \otimes \mathcal{O}(X)$.

Then, for $x \in G$, $(gf)(x) = \sum_{i=1}^{n} a_i(g^{-1}) \otimes b_i(x)$ and

$V = \langle b_1, \dots, b_n \rangle_k$ does the job.

(2) Choose generators $f_1, \dots, f_n$ of $\mathcal{O}(X)$. Via (1),

there is a f.d. G-stable $V \subset \mathcal{O}(X)$, s.t.

$G f_i \subset V$. Since V generates $\mathcal{O}(X)$ we obtain

a G-linear map $\mathrm{Sym}\, V = \mathcal{O}(V^*) \twoheadrightarrow \mathcal{O}(X)$ which

induces a closed immersion $X \to V^*$ $\square$

Thm Let G be an affine alg. group. Then G is linear.

Pf: By Lemma, we may find $f_1, \dots, f_n \in \mathcal{O}(G)$ that generate $\mathcal{O}(G)$ as a ring and the k-span V is G-stable.

WLOG assume that $f_1, \dots, f_n$ are linearly independent. Then we obtain from the action $\varphi: G \times V \to V$ a map $G \to GL(V) \cong GL_n(k)$ which can be explicitely described in the following way. We have

$$\varphi^\# f_i = \sum m_{ij} \otimes f_j$$

for $m_{ij} \in \mathcal{O}(G)$. This defines a map

$$\mathcal{O}(GL_n) = k[x_{ij}, \det^{-1}] \to \mathcal{O}(G), \quad x_{ij} \mapsto m_{ij}.$$

Since $f_i = \sum f_j(e) m_{ij}$, the m_{ij} generate $\mathcal{O}(G)$, so that the map is surjective, hence $G \hookrightarrow GL_n$ is a closed immersion $\square$

From now, we use linear and affine interchangeably.

1.5 Lie Algebras

<u>Recall</u>: Let A be a commutative k-algebra and M an A-module, then a k-linear map $\delta : A \to M$ satisfying the Leibniz rule $\delta(ab) = a\delta(b) + \delta(a)b$ is called a derivation. The set of derivations is denoted

$$\mathrm{Der}_k(A, M) \subset \mathrm{Hom}_k(A, M).$$

<u>Example</u>: (1) If X is an affine variety and $x \in X$, then we obtain a map $ev_x : \mathcal{O}_X \to \mathcal{O}_{X,x} = k$ which makes k an $\mathcal{O}_X$-module. Then $\mathrm{Der}_k(\mathcal{O}(X), \mathcal{O}(X)/m_x)$ $\cong T_x X$ is the set of point derivations at x.

(2) The set $\mathrm{Der}_k(A) = \mathrm{Der}_k(A, A) \subset \mathrm{End}_k(A)$

is closed under the Lie bracket $[\partial, \partial'] = \partial\partial' - \partial'\partial$

and hence a Lie algebra

(3) A map of A-modules $M \to N$ induces a map

$\mathrm{Der}_k(A, M) \to \mathrm{Der}(A, N)$

<u>Definition</u>: Let G be an affine algebraic group

Then the Lie-algebra of G is the set of k-linear, left invariant derivations

$$\mathfrak{g} = \mathrm{Lie}\, G \cong \mathrm{Der}_k(\mathcal{O}_G)^G = \{D \in \mathrm{Der}_k(\mathcal{O}_G) \mid \lambda_g \circ D = D\}$$

<u>Remark</u>: In other words, a derivation $\delta \in \mathrm{Der}_k(\mathcal{O}(G))$ is left-invariant if the following diagram commutes

$$\begin{tikzcd} \mathcal{O}(G) \arrow[r, "D"] \arrow[d, "\Delta"'] & \mathcal{O}(G) \arrow[d, "\Delta"] \\ \mathcal{O}(G)\otimes\mathcal{O}(G) \arrow[r, "id\otimes D"] & \mathcal{O}(G)\otimes\mathcal{O}(G) \end{tikzcd}$$

<u>Thm</u>: The following maps are inverse isomorphisms

$$\mathrm{Der}_k(\mathcal{O}(G))^G \underset{(2)}{\overset{(1)}{\rightleftarrows}} \mathrm{Der}_k(\mathcal{O}(G), \underbrace{\mathcal{O}(G)/m_e}_{=k})$$

(1) $D \mapsto ev_e \circ D$

(2) $\delta \mapsto (id\otimes\delta)\circ\Delta$

<u>Proof</u>: let $\delta \in \mathrm{Der}_k(\mathcal{O}(G), \mathcal{O}(G)/m_e)$, and denote

$D = (\mathrm{id} \otimes \delta) \circ \Delta$. We show that $D \in \mathrm{Der}_{\mathbb{R}}(\mathcal{O}(G))$.

For this, let $f, h \in \mathcal{O}(G)$. Then

$$D(fh) = (\mathrm{id} \otimes \delta)(\Delta(fh))$$

$$= (\mathrm{id} \otimes \delta)(\Delta(f)\Delta(h))$$

$$= (\mathrm{id} \otimes ev_e)(\Delta(f)) \cdot (\mathrm{id} \otimes \delta)(\Delta(h))$$

$$+ (\mathrm{id} \otimes ev_e)(\Delta(h)) \cdot (\mathrm{id} \otimes \delta)(\Delta(f))$$

$$= f \cdot D(h) + h D(f)$$

Similarly, one checks that D is left-invariant.

Exercise: Check that (1) + (2) are inverse $\square$

Hence, we see that we can identify

$$\mathrm{Lie}(\mathcal{G}) \cong T_e \mathcal{G}.$$

One shows that for a homomorphism of groups,

$\varphi : \mathcal{G} \to \mathcal{G}'$,

$$d\varphi : T_e \mathcal{G} \to T_e \mathcal{G}'$$

is indeed a map of Lie algebras.

<u>Example:</u> (1) Let $\mathcal{G} = GL_n$. Then

$$\mathrm{Lie}\, GL_n \cong \mathfrak{gl}_n \; (= (k^{n \times n}, [\,,\,])),$$

where $\delta \in \mathrm{Der}_k(\mathcal{O}(\mathcal{G}), \mathcal{O}(\mathcal{G})/m_e)$ corresponds to the matrix $(\delta(x_{ij}))_{i,j} \in k^{n \times n}$.

In a coordinate free description,

$$\mathrm{Lie}(GL(V)) = \mathfrak{gl}(V) = (\mathrm{End}_k(V), [\,,\,])$$

(2) If $\varphi: G \to G'$ is a closed immersion, then

$$d\varphi: \mathrm{Lie}(G) \hookrightarrow \mathrm{Lie}(G')$$ is injective.

In particular, if $G \subset GL_n$ is a linear algebraic group, then $\mathfrak{g} = \mathrm{Lie}\, G \subset \mathrm{Lie}\, GL_n = \mathfrak{gl}_n$.

(3) To compute the tangent space, it is often very useful to use the dual numbers. Namely, we get a short exact sequence of groups:

$$0 \to (T_1 G, +) \to G(k[\varepsilon]/\varepsilon^2) \xrightarrow{\varepsilon \mapsto 0} G(k) \to 1$$

This is especially useful for closed subgroups of GL_n. For example, consider

$$O_n = \{A \in GL_n \mid A A^{tr} = I_n\}.$$

Then, we obtain

$$\mathfrak{o}_n = \mathrm{Lie}\, O_n = \{X \in \mathfrak{gl}_n \mid \underbrace{(I_n + \varepsilon X)(I_n + \varepsilon X)^{tr}}_{\substack{\| \\ I_n + \varepsilon(X + X^{tr})}} = I_n\}$$

$$= \{X \in \mathfrak{gl}_n \mid X + X^{tr} = 0\}$$

Also, consider $SL_n = \{A \in GL_n \mid \det(A) = 1\}$, then

$$\mathfrak{sl}_n = \mathrm{Lie}\, SL_n = \{X \in \mathfrak{gl}_n \mid \underbrace{\det(I_n + \varepsilon X)}_{\substack{\| \\ 1 + \varepsilon \operatorname{tr}(X)}} = 1\}$$

$$= \{X \in \mathfrak{gl}_n \mid \operatorname{tr} X = 0\}$$

(4) For $g \in G$, let $c_g : G \to G$, $x \mapsto g^{-1} x g$.

Then we define $Ad(g) = dc_g \in GL(\mathrm{Lie}(G))$ and

Moreover $ad = dAd : \mathrm{Lie}(G) \to gl(\mathrm{Lie}(G))$

the <u>adjoint representations</u>.

For $G \subset GL_n$, $\mathrm{Lie}\, G \subset gl_n$, we get

$$Ad(g)X = gXg^{-1}, \quad ad(X)Y = [X, Y] \quad \square$$

1.6. Quotients

<u>Lem</u>: (Chevalley) Let G be lin. alg. $H < G$ closed subgroup. Then there is a G-rep. V and f.dim. subspace $L \subset V$, s.t., $G_L = \{g \in G \mid gL = L\} = H$

<u>Proof</u>: We have $H = \{g \in G \mid gH = H\}$.

Let $0 \to I(H) \to \mathcal{O}(G) \to \mathcal{O}(H) \to 0$. Then $H = G_{I(H)}$

Choose f.d. subspaces $U \subset I(H)$ and $U \subset W \subset \mathcal{O}(G)$, s.t., $\mathcal{O}(G)U = I(H)U$, $HU = U$, $GW = W$. Then $H = G_U$. Now, pass to $L = \Lambda^{\dim U} U \subset \Lambda^{\dim U} W = V$.

<u>Exercise</u>: $H = G_L$ $\square$

<u>Def</u>: Let G be an alg. grp., X a G-variety. A G-morphism $\pi: X \to Y$ is called a geometric quotient if

(1) π is a set-theoretical quotient $X \to X/G$

(2) π is open

(3) If $U \subset Y$ is open, then $\pi^{\#}: \mathcal{O}_Y(U) \to \mathcal{O}_X(\pi^{-1}(U))^G$ is an isomorphism.

Thm: let $H < \mathcal{G}$ be closed subgroup of an affine alg. grp.

Then there is a geometric quotient $\pi : \mathcal{G} \to \mathcal{G}/H$, s.t.

$\mathcal{G}/H$ is smooth and quasi-projective.

If H is normal in $\mathcal{G}$, then $\mathcal{G}/H$ is affine

Pf: By Chevalley's lemma, there is a $\mathcal{G}$-rep. V, s.t.

$H = \mathcal{G}_x$ for $x = [L] \in \mathbb{P}(V)$. Obtain a map:

$\pi : \mathcal{G} \to X = \mathcal{G} \cdot x \subset \mathbb{P}(V)$. This clearly satisfies (1).

To show (1) + (2) one uses Zariski's main thm and

faithfully flat descend $\square$

One may show that there is a s.e.s.

$$1 \to T_e H \to T_e G \longrightarrow T_e G/H \to 1$$

.

1.7 Jordan-Chevalley

Let $X \in \mathfrak{gl}_n$. Then, the Jordan decomposition, yields unique $X_n, X_s \in \mathfrak{gl}_n$, s.t.:

(1) $X_n + X_s = X$ (2) $X_n X_s = X_s X_n$

(3) X_s is diagonalizable (4) X_n is nilpotent

(5) $X_s = p(X)$, $X_n = q(X)$, for polynomials p, q.

Similarly, for $x \in GL_n$, we obtain $x_u = 1 + x_s^{-1} x_n$ where x_u is unipotent and $x_s x_u = x_u x_s = x$

If $x \in GL_n$ is unipotent we obtain a closed map

$$\mathbb{G}_a \hookrightarrow GL_n, \quad a \mapsto \exp(an)$$

Thm (Jordan - Chevalley): Let $\mathcal{G}$ be an affine alg. grp.

Then $x \in \mathcal{G}$, $X \in \mathfrak{g}$ admit intrinsic Jordan

decompositions $x = x_s x_u$, $X = X_s + X_n$ with $x_s, x_u \in \mathcal{G}$,

$X_s, X_u \in \mathfrak{g}$, compatible with homomorphism $\mathcal{G} \to H$

Pf: Omitted $\square$

Exercise: Show that semisimple/unipotent elements

in SL_2 do not form a subgroup.

1.8. Diagonalizable groups

<u>Def</u>: An affine alg. group D is called diagonalizable if $D \hookrightarrow (\mathcal{G}_m)^n$ or D is commutative and $d = d_s$ $\forall d \in D$.

If $D \cong (\mathcal{G}_m)^n$ is called a torus. $\square$

<u>Def</u>: For an algebraic group $\mathcal{G}$, we define

$$X(\mathcal{G}) = \mathrm{Hom}_{grp}(\mathcal{G}, \mathcal{G}_m), \quad Y(\mathcal{G}) = \mathrm{Hom}_{grp}(\mathcal{G}_m, \mathcal{G})$$

the character and cocharacter lattice. There is a pairing $\langle\, , \rangle : X(\mathcal{G}) \times Y(\mathcal{G}) \to \mathrm{Hom}_{grp}(\mathcal{G}_m, \mathcal{G}_m) \cong \mathbb{Z}$.

$X(\mathcal{G})$ is a group. If $\mathcal{G}$ is commutative, $Y(\mathcal{G})$ is a group and $\langle\, , \rangle$ bilinear $\square$

Exercise: (1) Prove that $\mathrm{Hom}_{grp}(\mathbb{G}_m, \mathbb{G}_m) \cong \mathbb{Z}$

(2) Let $T = (\mathbb{G}_m)^n$. Compute $X(T)$, $Y(T)$ and $\langle , \rangle$

(3) Let $G = \mu_n = \ker(x \mapsto x^n, \mathbb{G}_m \to \mathbb{G}_m)$. Show that μ_n is diagonalizable. Compute $X(\mu_n)$, $Y(\mu_n)$ and $\langle , \rangle$ $\boxtimes$

Thm: There is an equivalence of categories

$$\{\text{diagonalizable grp's}\} \longrightarrow \{\text{f.g. abelian grp's}\}^{op}$$

$$D \longmapsto X(D)$$

$$\text{Variety associated to the group algebra } k[X] \longleftarrow\!\shortmid \; X$$

$\square$

0.7. Solvable groups

<u>Def</u>: A group G is called solvable, if its derived series

$$G \supset [G,G] \supset [[G,G],[G,G]] \supset \dots$$

terminates in $\{e\}$ in f.t. many steps

<u>Example</u>: (1) Commutative $\Rightarrow$ solvable

(2) Subgroups and quotients of G are solvable if G is

(3) $B_n = \left\{ \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix} \right\} \subset GL_n$ is solvable (<u>Exercise</u>)

Thm (Lie-Kolchin+) (1) Let G be connected and solvable acting on a complete variety X. Then $X^G = \{x \in X \mid gx = x \ \forall g \in G\} \neq \emptyset$

(2) A connected solvable aff. alg. grp. G is of the form

$$G = \underset{\substack{\uparrow \\ \text{max. torus}}}{T} \ltimes \underset{\substack{\nwarrow \\ \text{unipotent elements in } G}}{G_u}$$

T is unique up to conjugation $\square$

2. Reductive groups and flag varieties

2.1. Borel subgroups and maximal tori

<u>Def</u>: Let G be an aff. alg. grp.

A Borel subgroup (maximal torus) of G is a maximal connected (semisimple) solvable subgroup of G. $\square$

<u>Thm</u>: Let $B \subset G$ be a Borel subgroup.
All Borel subgroups of G are conjugate to B and G/B is a projective variety.

Let $H \subset \mathcal{G}$ be a Borel subgroup of maximal dimension.

Let $L \subset V$ be as in 0.3. Lemma. Then H acts

on $\mathcal{F}l(V/L)$, which is projective (0.8 Remark).

Hence there is a fixed point $F' = (V_2/L \subset \dots \subset V_n/L)$, which

yields a flag $F = (V_1 = L \subset V_2 \dots \subset V_n) \in \mathcal{F}l(V)$, s.t.

$\mathcal{G}_F = H$ and $\mathcal{G}/H \to \mathcal{G}.F$ is bij.

Since the stabiliser of any flag is solvable and H maximal,

$\mathcal{G}.F$ has minimal dimension and is hence closed.

So $\mathcal{G}/H$ is projective (This needs an AG argument).

Now B acts on $\mathcal{G}/H$ and has a fixed point, say gH.

Hence $BgH = gH \iff gHg^{-1} \subset B$. This is an equality since both are Borel subgroups $\square$

Moreover we have the following facts, some closely related to the Theorem, which we will not prove here:

<u>Facts</u>: (1) For P parabolic $N_G(P)=P$

(Normalizer Thm.)

(2) If $T\subset B\subset G$ is a maximal torus in a Borel subgroup, then $C=C_G(T)\subset B$ and $N_B(T)=C_B(T)$

Moreover $C=T\times C_u$ and $\bigcup_{g\in G} g^{-1}Cg\subset G$ is dense

(3) If $D\subset G$ is diagonalizable, then $N_G^\circ(D)=C_G^\circ(D)$

(Rigidity of tori)

(4) If $H\subset G$ and $D\subset N_G(H)$ normalizes H, then

$\mathrm{Lie}(C_G(H)) = \mathfrak{c}_{\mathfrak{g}}(H)=\{X\in\mathfrak{g} \mid \mathrm{Ad}(h)(X)=X\ \forall h\in H\}$

(Infinitesimal = global centralizer)

Cor: (1) All maximal tori T in G are conjugate

Call $\operatorname{rank} G = \dim T$

(2) G/P projective $\Leftrightarrow$ P contains a Borel

Call P parabolic $\square$

(3) The following map is an isomorphism

$$G/B \longrightarrow \mathcal{B} = \{B' \subset G \mid B' \text{ Borel subgroup}\}$$

$$gB \longmapsto gBg^{-1}$$

Call $\mathcal{B}$ the full flag variety associated to G.

(4) If $\varphi: G \to G'$ is surjective, then it preserves

maximal tori, Borel subgroups, parabolics.

Example: Consider $B_n = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\} \subset GL_n$. Then

$GL_n / B_n \xrightarrow{\sim} Fl(k^n)$ is projective, so B_n is

parabolic and solvable. Hence B_n is a Borel subgroup.

2.2. Weyl group – Basics

<u>Def</u>: For $H \subset G$, $W(G,H) = N_G(H)/C_G(H)$ is called the Weyl group of H in G. $\square$

Clearly, $W(G,H)$ only depends on the conjugacy class of H in G. For an affine alg. group G, the Weyl group is $W = W(G) = W(G,T)$ for a maximal torus $T \subset G$.

<u>Rem.</u>: By Fact 2.1. (3) $W(G,H)$ is finite if H is any diagonalizable subgroup of G. So $W = W(G)$ is finite.

Proposition: Let T be a maximal torus. Then

Then $\mathcal{B}^T$ is a $W(G,T)$-torsor.

Proof: Note that $T \in \mathcal{B}^T \Leftrightarrow T \subset N_G(B) = B$, where

we used Fact 1.1.(1). By Fact 1.1.(2) $C_G(T) \subset B$ for

if $T \subset B$. Hence, the action of $W(G,T)$ on $\mathcal{B}^T$ is well-def..

Transitivity: Let $B, B' \in \mathcal{B}^T$. Then there is $x \in G$ with

$B' = xBx^{-1}$ (Thm. 1.1). Then $T, xTx^{-1} \subset B'$ are maximal tori,

so there is $y \in B'$ with $T = yxTx^{-1}y^{-1}$. Hence $z = yx \in N_G(T)$

and $zBz^{-1} = B'$

Freeness: Let $x \in N_G(T)$, $B \in \mathcal{B}^T$, s.t. $xBx^{-1} = B$.

So $x \in N_{\mathcal{G}}(B) = B$. Hence $x \in B \cap N_{\mathcal{G}}(T) = N_B(T) = C_{\mathcal{G}}(T)$

where we used Fact 1.1.(2). So, $e = [x] \in W(\mathcal{G}, T)$ $\square$

In particular, $\mathcal{B}^T$ is finite.

<u>Fact</u>: Let $\varphi: \mathcal{G} \to \mathcal{G}'$ be surjective, $T \subset \mathcal{G}$, $T' = \varphi(T) \subset \mathcal{G}'$

max. tori. Then $\mathcal{B}^T \to \mathcal{B}'^{T'}$ and $W(\mathcal{G}, T) \to W(\mathcal{G}', T')$

are surj. Moreover, if $\ker \varphi \subset \bigcap_{B \in \mathcal{B}} B$, then these

maps are isomorphisms $\square$

2.3. Summoning Fixed Points

Lemma: Let $\mathbb{G}_m$ act on k^n via $\lambda(a_i) = (\lambda^{m_i} a_i)$.

and $v = (a_i) \in k^n$. Denote

$$I = \{ i \in I \mid a_i \neq 0 \}, \quad m_{min} = \min_{i \in I} m_i, \quad m_{max} = \max_{i \in I} m_i$$

$$I_{min} = \{ i \in I \mid m_i = m_{min} \}, \quad I_{max} = \dots$$

$$v_{min} = \sum_{i \in I_{min}} a_i e_i \qquad v_{max} = \dots$$

Then, the action of $\mathbb{G}_m$ on $[v] \in \mathbb{P}(k^n)$ can be extended

$$\begin{tikzcd}
\lambda \arrow[r, mapsto] & {[\lambda v]} \\
\mathbb{G}_m \arrow[r] \arrow[d] & \mathbb{P}(k^n) \\
\mathbb{P}^1 \arrow[ur] &
\end{tikzcd}$$

via $\quad 0 \mapsto [v_{min}], \quad \infty \mapsto [v_{max}]$

Moreover $[v_{min}]$ and $[v_{max}]$ are fixed points which are distinct if $[v]$ is not a fixed point.

Proof: We have

$$\lambda[a_i] = [\lambda^{m_i} a_i] = [\lambda^{m_i - m_{min}} a_i] = [\lambda^{m_i - m_{max}} a_i]$$

So, since $m_i - m_{min} \geq 0$ and $m_i - m_{max} \leq 0$, the expression is well-defined for $\lambda = 0, \infty$, and yields $[v_{min}]$ and $[v_{max}]$, respectively.

Thm: In the notation of the lemma, let $X \subset \mathbb{P}(k^n)$ be an irreducible closed subset.

(a) If $\dim X > 1$, $|X^{\mathbb{G}_m}| \geq 2$

(b) If $\dim X > 2$, and $X \not\subset \mathbb{P}(W)$ for all $W \subset k^n$ with $\dim W = n-1$, $|X^{\mathbb{G}_m}| \geq 3$

Proof: (a) Since $\dim X > 1$, $|X| = \infty$. If $X^{\mathbb{G}_m} = \mathbb{G}_m$ we are done. Else, let $v \in X - X^{\mathbb{G}_m}$. We obtain, as in lemma,

$$\begin{tikzcd}
\mathbb{G}_m \arrow[r, "(*)"] \arrow[d] & X \arrow[r] & \mathbb{P}(k^n) \\
\mathbb{P}^1 \arrow[ur, "(**)"] \arrow[urr] & &
\end{tikzcd}$$

where $(*)$ exists since X is $\mathbb{G}_m$-stable and $(**)$ since X is closed. $[v_{min}], [v_{max}] \in X$ are the desired two fixed points.

(b) Proceed as in (a), assume that $[v] \in X - X^{\mathbb{G}_m}$. Choose

$i^* \in I_{max}$ and let $W = \langle e_i \mid i \neq i^* \rangle$.

Then by general AG fact, irreducible components of

$X \cap \mathbb{P}(W)$ have codim. 1 in X. Choose one of them,

say Y. By induction $|Y^{\mathbb{G}_m}| = 2$.

Now $[v_{max}] \in X - \mathbb{P}(W)$ is the third fixed point $\square$

<u>Corollary</u>: (1) Let $P \subsetneq G$ be a parabolic subgroup in

a lin. alg. grp., $T \subset G$ any torus. Then $|(G/P)^T| \geq 2$

(or ≥ 3 if $\dim G/P > 1$)

(2) If $\dim \mathcal{B} > 1$, then $|\mathcal{B}^T| = |W| \geq 2$.

(3) If $T \subset G$ is a maximal torus, then

G is generated by $\mathcal{B}^T$

<u>Proof</u>: (1) Sketch: Choose G-rep. V, such

that $G/P \xrightarrow{\sim} X \subset \mathbb{P}(V)$ equivariantly. Wlog.

$X \not\subset \mathbb{P}(W)$ for any $W \subsetneq V$. Now, choose cocharacter

$\lambda: \mathbb{G}_m \to T$, such that $X^T = X^{\mathbb{G}_m}$. Then

apply Theorem

(2) Clear (3) Omitted $\square$

2.4. Reductivity / Semisimplicity

Def + Lemma: Let G be an connected aff. alg grp. Then G contains a unique largest normal connected (unipotent) solvable subgroups

$$R_u(G), R(G) \trianglelefteq G$$

called the (unipotent) radical of G.

G is called reductive or semisimple if $R_u(G)$ or $R(G)$ are trivial, resp.

Sketch:

$$\overbrace{R_u(G) \subseteq R(G^\circ)}^{\text{torus}} \subseteq \overbrace{G^\circ \subseteq G}^{\text{finite}}$$

$$R_u(G) \subseteq \underbrace{R(G^\circ) \subseteq G^\circ}_{\text{semisimple}} \subseteq G$$

$$\underbrace{R_u(G) \subseteq R(G^\circ) \subseteq G^\circ}_{\text{reductive}} \subseteq G$$

<u>Remark</u>: (1) One may show that:

$$R(G) = \bigcap_{B \in \mathcal{B}} B \quad \text{and} \quad R_u(G) = \bigcap_{B \in \mathcal{B}} B_u \,.$$

(2) The semi-simple (reductive) rank of G is

$$\operatorname{rank}_{ss} G = \operatorname{rank} G/R(G) \quad \text{and} \quad \operatorname{rank}_{red} G/R_u(G) \quad \square$$

<u>Thm</u>: Let G be a connected lin. alg. grp., $T \subset G$ max. torus and $W = W(G,T)$. Then t.f.s.a.e.:

(a) $\operatorname{rank}_{ss} G = 1$, (b) $|W| = |\mathcal{B}^T| = 2$, (c) $\dim \mathcal{B} = 1$

(d) $\mathcal{B} \cong \mathbb{P}^1$ (e) There is an epi, $q\; G \to PGL_2$ with

$(\ker q)^\circ = R(G)$

Proof: Exercise, use Corollary 1.3, $\mathrm{Aut}_{grp}\, \mathbb{G}_m = \mathbb{Z}/2$ and $\mathrm{Aut}(\mathbb{P}^1) = PGL_2$ $\square$

Cor A: Let G be reductive, $\mathrm{rank}_{ss}\, G = 1$, $T \subset G$ max. torus, $Z = Z(G)$. Then

(a) $G' = [G, G]$ is semisimple of dim 3

(b) $G = G'Z$, $G' \cap Z$ finite.

(c) $C_G(T) = T$, $Z(G) \subset T$

(d) If $\varphi: G \to PGL_2$ is epi, then $\ker \varphi = Z(G)$ $\square$

2.5. Weights and Roots

Let $T \cong \mathbb{G}_m^n$ be a torus

<u>Def</u>: (1) Let V be a T-rep., $\chi \in X(T)$ and

$$V_\chi = \{ v \in V \mid tv = \chi(t) v \ \forall t \in T \}$$

Moreover $\Lambda(V) = \{ \chi \mid V_\chi \neq 0 \}$ is the set of <u>weights</u>

$$V = \bigoplus_\chi V_\chi$$

is called the weight space decomposition of V

(2) Let $T \subset G$ be a maxi .

T acts on $\mathfrak{g}$ via Ad Then $\Phi(G,T) = \Lambda(\mathfrak{g}) - \{0\}$

is called the set of <u>roots</u> of G, relative to T

So we get

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Phi(\mathfrak{g},T)} \mathfrak{g}_\alpha$$

where $\mathfrak{g}_0 = c_{\mathfrak{g}}(T) = \mathrm{Lie}(C_G(T))$

<u>Remark</u>: (1) It will also be useful to consider the following set of reduced roots: $\Phi_{red}(\mathfrak{g},T) \subset \Phi(\mathfrak{g},T)$, which is defined via

$$\mathfrak{g} = \mathrm{Lie}\Big(\bigcap_{\substack{T \subset B \\ B \subset G \\ \text{Borel}}} B\Big) \oplus \bigoplus_{\alpha \in \Phi_{red}(\mathfrak{g},T)}$$

(2) The Weyl group $W = W(\mathcal{G},T) = N_{\mathcal{G}}(T)/C_{\mathcal{G}}(T)$ acts on $X(T)$ and $Y(T)$, where $w \in N_{\mathcal{G}}(T)$ acts via

$$(w\chi)(t) = \chi(w^{-1}tw), \text{ for } \chi \in X(T), t \in T$$

$$(w\lambda)(z) = w\lambda(z)w, \text{ for } \lambda \in Y(T), z \in \mathbb{G}_m.$$

This action is compatible with $\langle\, , \rangle$, i.e, $\langle w-,-\rangle = \langle -, w-\rangle$.

Moreover, the action descends to $\Phi \subset X(T)$ $\square$

<u>Exercise</u>: Compute $\Phi \subset X(T)$ and the action of W for $\mathcal{G} = GL_n$ $\square$

Cor: Let G be reductive with $\mathrm{rank}_{ss}\, G = 1$.

T a max. torus in G. Then

(1) $\ker(Ad) = Z(G)$,

(2) $\Phi = \Phi_{red} = \{\pm\alpha\}$ for some $\alpha \in \Phi$.

(3) $\mathcal{B}^T = \{B_\alpha, B_{-\alpha}\}$, s.t. for $U_{\pm\alpha} = (B_{\pm\alpha})_u$,

$$\mathfrak{g}_{\pm\alpha} = \mathrm{Lie}\, U_{\pm\alpha}$$

(4) There is an isomorphism

$$u_\alpha : \mathfrak{g}_\alpha \longrightarrow U_\alpha$$

(5) $$\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$$ $\square$

<u>Example</u>: $G = GL_2 \supset T = \{(\diagdown)\}$.

$R(G) = Z(G) = \{\begin{pmatrix} x & \\ & x \end{pmatrix}\}$, $GL_2/Z(G) = PGL_2$

$\varepsilon_i : T \to \mathbb{G}_m \quad \begin{pmatrix} z_1 & \\ & z_2 \end{pmatrix} \mapsto z_i$

$\alpha = \varepsilon_1 - \varepsilon_2 : \quad \begin{pmatrix} z_1 & \\ & z_2 \end{pmatrix} \mapsto z_1 z_2^{-1}$

$$\begin{array}{ccccc} \mathfrak{g} = \mathfrak{g}_0 & \oplus & \mathfrak{g}_\alpha & \oplus & \mathfrak{g}_{-\alpha} \\ \| & & \| & & \| \\ \{\begin{pmatrix} a & \\ & d \end{pmatrix}\} & & \{\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}\} & & \{\begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix}\} \\ \| & & \| & & \| \\ \mathrm{Lie}\, T & & \mathrm{Lie}\, U_\alpha & & \mathrm{Lie}\, U_{-\alpha} \end{array}$$

$B_\alpha = \{\begin{pmatrix} * & * \\ & * \end{pmatrix}\}$, $B_{-\alpha} = \{\begin{pmatrix} * & \\ * & * \end{pmatrix}\}$

$U_\alpha = \{\begin{pmatrix} 1 & b \\ & 1 \end{pmatrix}\}$, $U_{-\alpha} = \{\begin{pmatrix} 1 & \\ c & 1 \end{pmatrix}\}$

$u_\alpha : \mathbb{G}_a \to U_\alpha$, $x \mapsto \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}$

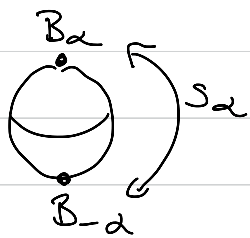
$$W = N_{\mathfrak{g}}(T)/T = \left\{ \begin{pmatrix} x & \\ & y \end{pmatrix}, \begin{pmatrix} & x \\ y & \end{pmatrix} \right\} / T$$

$$\downarrow\wr$$

$$S_2 = \{ e \;, \; s_\alpha \}$$

$$s_\alpha(\alpha) = -\alpha \;, \quad s_\alpha B_\alpha s_\alpha = B_{-\alpha}$$

$$\mathcal{B} \cong \mathbb{P}^1$$



2.6. Summoning PGL_2's

<u>Definition</u>: Let $S \subset G$ be a torus in a conn. lin. alg. grp. Then S is called regular, if $\mathcal{B}^S$ is finite. Otherwise S is called singular. $\square$

<u>Proposition</u>: Let $S \subset T \subset G$ be a torus and maximal torus in G. Then S is regular iff $C = C_G(S)$ is solvable. In this case $\mathcal{B}^T = \mathcal{B}^S$ $\square$

<u>Proof</u>: Let $Y \subset X = \mathcal{B}^S$ be a connected component. Note that C is connected (If $x \in C$, $s \in S$, both lie in a common Borel subgroup, which is connected). Hence, C acts on Y.

<u>Claim:</u> C acts trasitively on Y.

Let $B \in Y$ and $Z = \{g \in G \mid gBg^{-1} \in Y\}$

We obtain a map

$$q: Z \times S \to B \to B/B_u = H, \quad (z,s) \mapsto zsz^{-1}.$$

where H is also a torus. By the following Lemma q is constant along Z. It follows that $zSz^{-1} \subset SB_u$ for all $z \in Z$. So, there is a $u \in B_u$, s.t. $nSn^{-1} = S$, for $n = uz \in N_G(S)$. So $CB \subset Z < NB$.

Since N/C is finite, $\dim CB = \dim Z$. Both are connected, so $CB = Z$. Hence $C \cdot B = Z$ //Claim

We obtain $C/C\cap B \xrightarrow{\sim} Y \subset \mathcal{B}$ is complete

So $C\cap B \subset C$ is a Borel subgroup in C.

Hence :

$$C \text{ solvable} \iff C\cap B = C \iff \dim \mathcal{B}^S = 0 \iff \mathcal{B}^S \text{ finite}$$

The rest follows similarly $\square$

<u>Corollary</u>: S is singular iff $S \subset T_\alpha$ for some $\alpha \in \Phi_{red}$.

<u>Proof</u> Omitted. $\square$

<u>Lemma</u>: Let $\varphi : Z \times S \to H$ be a morphism where

(1) $S_{fin} = \{ s \in S \mid |s| < \infty \} \subset S$ is dense

(2) $H_m = \{ h \in H \mid h^m = 1 \}$ is finite for each $m \in \mathbb{Z}_{>0}$

(3) Z is connected

(4) For all $z \in Z$, $\varphi(z, -) : S \to H$ is a group hom.

Then φ is constant along Z.

<u>Pf</u>: Let $s \in S$ with $s^m = 1$. Then φ restricts to

$Z \times \{s\} \to H_m$, Since Z is connected, the image

is a point. Hence $Z \times S_{fin} \to H$ is constant

along Z. The statement follows from (1) $\square$

Thm: Let $G \supset T$ be a conn. lin. alg. grp. with maximal torus. For $\alpha \in \Phi_{red}(G,T)$, let

- $T_\alpha = \ker(\alpha : T \to \mathbb{G}_m)^\circ$
- $Z_\alpha = C_G(T_\alpha)$
- $G_\alpha = Z_\alpha / R(Z_\alpha)$

Then G_α is a semisimple group of rank 1.

Proof: We have $T_\alpha \subset Z(Z_\alpha)^\circ \subset R(Z_\alpha)$.

Since Z_α is not solvable $R(Z_\alpha) \subsetneq Z_\alpha$ and

G_α has rank $= \dim T/T_\alpha = 1$ $\square$

2.7. Weyl chambers

Fix $G \supset T$ conn. lin. alg. grp. with max torus.

Let $\alpha \in \Phi_{red}$. Then, we obtain

$$\{e, s_\alpha\} = W(Z_\alpha, T_\alpha) = W_\alpha \subset W.$$

Moreover Z_α has two Borel subgroups $B_{-\alpha}, B_\alpha$,

s.t. $\mathfrak{g}_{\pm\alpha} = \mathrm{Lie}\, U_{\pm\alpha}$, $U_{\pm\alpha} = (B_{\pm\alpha})_u$.

We have $s_\alpha B_{-\alpha} s_\alpha = B_\alpha$.

Let $B \in \mathcal{B}^T$, then $B \cap Z_\alpha \in \{B_{-\alpha}, B_\alpha\}$, so

α induces a partition of $\mathcal{B}^T$.

Let $Y(T)_{reg} = \{\lambda : \mathbb{G}_m \to T \mid \lambda(\mathbb{G}_m) \text{ is regular}\}$

$$= \{\lambda \in \mathbb{G}_m \to T \mid \langle \alpha, \lambda \rangle \neq 0, \ \forall \alpha \in \Phi_{red}\}$$
$$(\Leftrightarrow \lambda(\mathbb{G}_m) \not\subset T_\alpha).$$

Let $\lambda \in Y(T)_{reg}$. Then, $\mathcal{B}^T = \mathcal{B}^{\lambda(\mathbb{G}_m)}$ and there is a unique maximal fixed point $B(\lambda) \in \mathcal{B}^{\lambda(\mathbb{G}_m)}$.

Here, maximal means that for all $B \in \mathcal{B}$, $\lambda(0)B \neq B(\lambda)$.

In fact $B(\lambda) \in \mathcal{B}^T$ is characterized by the property

$$B(\lambda) \cap Z_\alpha = B_\alpha \Leftrightarrow \langle \lambda, \alpha \rangle > 0.$$

For $B \in \mathcal{B}^T$, denote by

$$\mathcal{C}(B) = \{\lambda \in Y(T)_{reg} \mid B = B(\lambda)\}$$

Example: $GL_3 = G \supset T = \{(\ddots)\}$

$X(T) = \langle \varepsilon_i : \mathrm{diag}(z_1, z_2, z_3) \mapsto z_i \rangle$

$\Phi = \Phi_{red} = \{\pm\alpha, \pm\beta, \pm(\alpha+\beta)\}$, $\alpha = \varepsilon_1 - \varepsilon_2$, $\beta = \varepsilon_2 - \varepsilon_3$

$T_\alpha = \left\{\begin{pmatrix} z_1 & & \\ & z_1 & \\ & & z_3 \end{pmatrix}\right\} \rightsquigarrow Z_\alpha = \left\{\begin{pmatrix} * & * & \\ * & * & \\ & & * \end{pmatrix}\right\}$,

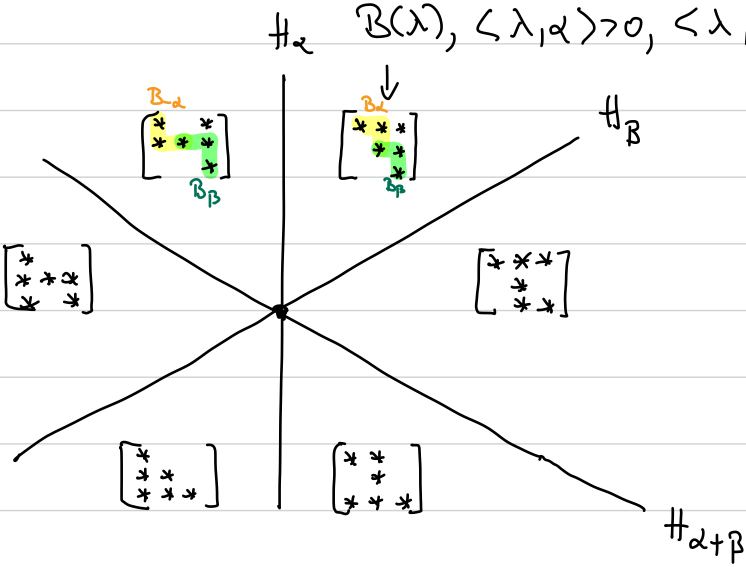
$R(Z_\alpha) = T_\alpha$, $G_\alpha = Z_\alpha / T_\alpha = \left\{\begin{pmatrix} \left[\begin{smallmatrix} * & * \\ * & * \end{smallmatrix}\right] & \\ & [*] \end{pmatrix}\right\} \cong PGL_2$

where [ ] indicates elements in PGL_2.

$B_\alpha = \begin{pmatrix} * & * & \\ & * & \\ & & * \end{pmatrix} \subset Z_\alpha$, $B_{-\alpha} = \begin{pmatrix} * & & \\ * & * & \\ & & * \end{pmatrix}$

$\rightsquigarrow$ See GL_2 example 1.4

$$H_\chi = \{\lambda \in Y(T) \mid \langle \chi, \lambda \rangle = 0\}$$



$$\bullet = H_\alpha \cap H_\beta = \langle \varepsilon_1^* + \varepsilon_2^* + \varepsilon_3^* \rangle \subset Y(T)$$
$$= Y(Z(g))$$

2.8. The unipotent radical

Recall that $I(T) = \left(\bigcap_{B \in \mathcal{B}^T} B\right)^\circ$.

<u>Thm</u>: Let $G \supset T$ be a conn. lin. alg. grp. with max. torus.

Then $R_u(G) = I(T)_u$

~~Pf~~: Let $U = I(T)_u$. Clearly $R_u(G) \subset U$.

So we need to show U is normal G.

Now $G = \langle B \in \mathcal{B}^T \rangle$, so it suffices to show $B \subset N_G(U)$.

Choose $B \in \mathcal{B}^T$. Let $A = \langle C_B(S) \mid S \subset T$ torus, $\dim T/S = 1 \rangle$

We claim that $A = B$. Suffices to show $\mathfrak{a} = \mathfrak{b}$. For this,

we compare $\mathfrak{a}_\lambda$ and $\mathfrak{b}_\lambda$ for $\lambda \in X(T)$.

If $\lambda = 0$, we have

$$\mathfrak{b}_0 = c_{\mathfrak{b}}(T) = \mathrm{Lie}(C_B(T)) \subset \mathrm{Lie}(C_G(S)) \subset \mathfrak{a}$$

For $\lambda \neq 0$, let $S = (\ker \lambda)^\circ$, then, similarly,

$$\mathfrak{b}_\lambda = c_{\mathfrak{b}}(S) \subset \mathfrak{a}.$$

WTS $U \trianglelefteq G \rightsquigarrow B \subset N_G(U)$,
$B = \langle C_B(S) \mid S \subset T,\ \dim T/S = 1 \rangle$
Suffices to show: $C_B(S) \subset N_G(U)$

So, it suffices to show $C_B(S) \subset N_G(U)$, $S \subset T$ torus, $\dim(T/S) = 1$. If S is regular, $C_B(S) = C_G(S)$ (1.5. Proposition) and $C_G(S)$ is contained in all Borel subgroups in $\mathcal{B}^S = \mathcal{B}^T$. (1.1 Facts)

So $C_B(S) \subset N_G(U)$.

Now, let S be singular, so $S = (\ker \alpha)^\circ = T_\alpha$ for $\alpha \in \Phi_{red}$

(1.5 Corollary). Then

$$C_B(T_\alpha) = B \cap C_G(T_\alpha) = B \cap Z_\alpha \in \{B_\alpha, B_{-\alpha}\}.$$

W.l.o.g. $C_B(T_\alpha) = B_\alpha$, so $B = B(\lambda)$ with

$\langle \alpha, \lambda \rangle > 0$ for some $\lambda \in Y_{reg}(T)$.

In total, we can sort the weights in $\mathfrak{g}$

$$\underbrace{-\alpha, -\beta, \dots}_{\mathfrak{g}/\mathfrak{b}}, \quad \underbrace{0}_{\mathfrak{t}}, \quad \underbrace{\alpha, \beta, \dots}_{\mathfrak{b}_u/\mathfrak{u}}, \quad \underbrace{\gamma, \delta, \dots}_{\mathfrak{u}}$$

$\in \Phi_{red}$

Let $H = \bigcap_{\langle \alpha, \lambda \rangle > 0} B(\lambda)_u \supset U$. Then $(B_\alpha)_u \subset H$.

Let β be a weight of $\mathfrak{h}/\mathfrak{u}$. Assume $\alpha \neq \beta$.

Then $\beta \neq 0$ and $\beta \neq -\alpha$ (Since $(B_{-\alpha})_u \not\subset B(\lambda)$ if

$\langle \alpha, \lambda \rangle > 0$). Now α, β are are non-proportional,

since else $T_\alpha = T_\beta$, $Z_\alpha = Z_\beta$ and $\Phi_{red}(Z_\alpha = Z_\beta) = \{\alpha, -\alpha\} = \{\beta, -\beta\}$ ↯

Choose a $\lambda \in Y(T)_{reg}$, s.t., $\langle \alpha, \lambda \rangle > 0$, $\langle \beta, \lambda \rangle < 0$

Then $B(\lambda) \cap Z_\beta = B_{-\beta}$ with $B(\lambda) \supset H$.

Sine $-\beta$ does not occur in $\mathfrak{h}$, we have

$Z_\beta \cap H = C_H(T_\beta) = 1$, so $C_{\mathfrak{h}}(T_\beta) = \mathfrak{h}_\beta = 0$ ↯

Hence $\mathfrak{h}/\mathfrak{u} = \mathfrak{g}_\alpha$ and $\dim H/U = 1$. Since

H is unipotent it is nilpotent. So $U \trianglelefteq H$. $\square$

<u>Corollary</u>: Let G be reductive, $T \subset G$ max. torus.

(a) For a torus $S \subset T \subset G$, $C_G(S)$ is reductive

and $C_G(S) = T$ if S is regular

(b) $\Phi = \Phi_{red}$, $\Phi = -\Phi$

(c) $\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$, $\dim \mathfrak{g}_\alpha = 1$

(d) Z_α is reductive and $\mathfrak{z}_\alpha = \mathfrak{t} \oplus \mathfrak{g}_{-\alpha} \oplus \mathfrak{g}_\alpha$,

the Z_α generate G.

(e) $Z(G) = \bigcap_{\alpha \in \Phi} T_\alpha$, $X(Z(G)) = X(T)/\mathbb{Z}\Phi$.

(f) For each $B \in \mathcal{B}$ there is a $B^- \in \mathcal{B}$ with

$B^- \cap B = T$, $\mathfrak{g} = \mathfrak{b}^- + = \mathfrak{u}^- \oplus \mathfrak{t} \oplus \mathfrak{u}$.

Proof: Sketch + Exercise

(a) Let $C = C_{\mathfrak{g}}(S)$. Then $R_u(C) = I(T)_u \cap C \subset I(T)_u = \{1\}$

so C is reductive.

If S is regular, C is solvable (2.6 Proposition).

But reductive + solvable = torus, so $C = T$.

(b), (d) immediate.

(e) $\bigcap T_\alpha \subseteq \bigcap C_{\mathfrak{g}}(Z_\alpha) \underset{\uparrow}{\subseteq} Z(\mathfrak{g}) \subset \bigcap Z(Z_\alpha) = \bigcap T_\alpha$

Z_α generate $\mathfrak{g}$

(f) Let $B = B(\lambda)$, set $B^- = B(-\lambda)$

Addendum to 2.6. Proposition.

Why is $C = C_G(S)$ connected for $S \subset G$ a torus in a connected group?

<u>Sketch</u>: Let $x \in C$. Then one shows that x is contained in a Borel subgroup B containing S (consider S acting on $\emptyset \neq \mathcal{B}^x \subset \mathcal{B}$ which has a fixed point)

Hence, we can assume that $G = B$ is solvable.

Then, take $s \in S$. One shows that $C_G(s) = T \ltimes C_G(s)_u$ for $S \subset T \subset G$ a maximal torus. Now $U = C_G(S)_u$ is unipotent, and hence connected: If $u \in U$, then

then there is a map $\mathfrak{g}_a \to U$, $x \mapsto \exp(x \log u)$

and hence $u \in U^0$ $\square$

2.9 Root subgroups, SL_2 and the root datum.

Thm: Let G be reductive with max. torus T, $\alpha \in \Phi = \Phi(G,T)$.

(a) There is a unique T-stable connected subgroup

$$U_\alpha < B_\alpha \subset Z_\alpha \subset G \quad \text{with } \operatorname{Lie} U_\alpha = \mathfrak{g}_\alpha.$$

(b) If $w = [n] \in W$, $n U_\alpha n^{-1} = U_{n(\alpha)}$.

(c) There is an isomorphism

$$u_\alpha : \mathbb{G}_a \to U_\alpha, \text{ s.t.}$$

$$t u_\alpha(x) t^{-1} = u_\alpha(\alpha(t) x)$$

(d) There is a $(2:1$ or $1:1)$ map

$$SL_2 \to [Z_\alpha, Z_\alpha] \to Z_\alpha \to G$$

such that

$$\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \mapsto u_\alpha(x) \qquad \begin{pmatrix} 1 & \\ y & 1 \end{pmatrix} \mapsto u_{-\alpha}(x)$$

This yields an isom. of Lie algebras.

$$\mathfrak{sl}_2 \to \mathfrak{sl}_\alpha = [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$$

<u>Def + Remark</u>: For $\alpha \in \Phi$, we obtain a unique cocharacter $\alpha^\vee : \mathbb{G}_m \to [Z_\alpha, Z_\alpha] \cap T \to T$, such that $\langle \alpha, \alpha^\vee \rangle = 2$. We call $\alpha^\vee$ the coroot associated to α and denote by

$$\underline{\Phi}^\vee = \{\alpha^\vee \mid \alpha \in \underline{\Phi}\} \subset Y(T)$$

the set of coroots.

To $G \supset T$ we hence associate the following data:

$$(X(T), \Phi, Y(T), \Phi^\vee, \langle \, , \, \rangle, {}^\vee : \Phi \xrightarrow{\sim} \Phi^\vee)$$

which we call the <u>root datum</u> associated to $G \supset T$.

Often $\langle \, , \, \rangle$ and the bijection ${}^\vee$ are dropped from the notation.

29. An extended example: Sp_4

Let $G = Sp_4 = \{ A \in GL_4 \mid A J_4 A^{tr} = J_4 \}$

where $J_4 = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{pmatrix}$.

Then we get

$$\mathfrak{g} = \mathfrak{sp}_4 = \{ X \in \mathfrak{gl}_4 \mid X J_4 + J_4 X^{tr} = 0 \}$$

$$= \left\{ \begin{pmatrix} A & B \\ C & -A^{tr} \end{pmatrix} \;\middle|\; B^{tr} = B,\ C^{tr} = C \right\}$$

and $\dim Sp_4 = 2(2)^2 + 2 = 10$

And the maximal torus

$$T = \left\{ \begin{pmatrix} x_1 & & & \\ & x_2 & & \\ & & x_1^{-1} & \\ & & & x_2^{-1} \end{pmatrix} \right\} \subset G$$

$X(T) = \langle \varepsilon_1, \varepsilon_2 \rangle$, $Y(T) = \langle \varepsilon_1^*, \varepsilon_2^* \rangle$.

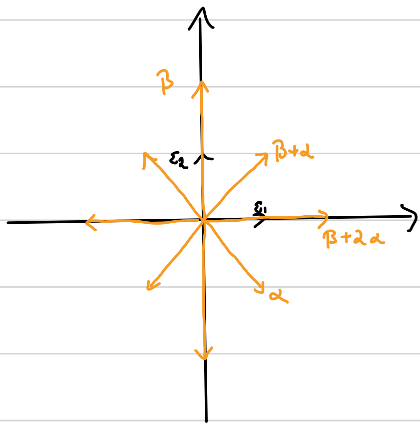
γ	U_γ	$\gamma^\vee(\mathbb{G}_m)$	$\gamma^\vee$
$\alpha = \varepsilon_1 - \varepsilon_2$	$\left(\begin{array}{cc\|cc} 1 & a & & \\ & 1 & & \\ \hline & & 1 & \\ & & -a & 1 \end{array}\right)$	$\left(\begin{array}{cc\|cc} x & & & \\ & x^{-1} & & \\ \hline & & x^{-1} & \\ & & & x \end{array}\right)$	$\alpha^\vee = \varepsilon_1^* - \varepsilon_2^*$
$\beta = 2\varepsilon_2$	$\left(\begin{array}{cc\|cc} 1 & & & \\ & 1 & & a \\ \hline & & 1 & \\ & & & 1 \end{array}\right)$	$\left(\begin{array}{cc\|cc} 1 & & & \\ & x & & \\ \hline & & 1 & \\ & & & x^{-1} \end{array}\right)$	$\beta^\vee = \varepsilon_2^*$
$\alpha+\beta = \varepsilon_1 + \varepsilon_2$	$\left(\begin{array}{cc\|cc} 1 & & & a \\ & 1 & a & \\ \hline & & 1 & \\ & & & 1 \end{array}\right)$	$\left(\begin{array}{cc\|cc} x & & & \\ & x & & \\ \hline & & x^{-1} & \\ & & & x^{-1} \end{array}\right)$	$(\alpha+\beta)^\vee = \varepsilon_1^* + \varepsilon_2^*$
$2\alpha+\beta = 2\varepsilon_1$	$\left(\begin{array}{cc\|cc} 1 & & a & \\ & 1 & & \\ \hline & & 1 & \\ & & & 1 \end{array}\right)$	$\left(\begin{array}{cc\|cc} x & & & \\ & 1 & & \\ \hline & & x^{-1} & \\ & & & 1 \end{array}\right)$	$(2\alpha+\beta)^\vee = \varepsilon_1^*$

and $U_{-\gamma} = U_\gamma^{tr}$. For example:

$$Z_\alpha = \langle T, U_\alpha, U_{-\alpha} \rangle = \left(\begin{array}{c|c} A & \\ \hline & A^{-tr} \end{array}\right) \qquad [Z_\alpha, Z_\alpha] = SL_2$$

$$Z_{\alpha+\beta} = \langle T, U_{\pm(\alpha+\beta)} \rangle = \left(\begin{array}{cc|cc} a & & & b \\ & a & b & \\ \hline & c & d & \\ c & & & d \end{array}\right) \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2$$

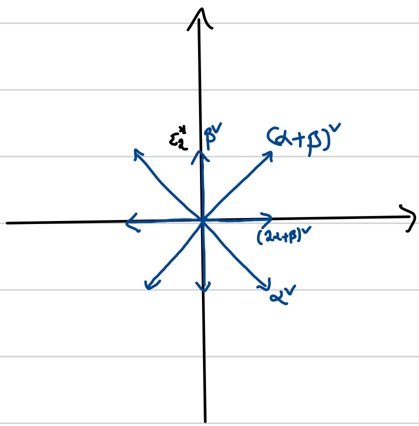
$\Phi \subset X(T)$



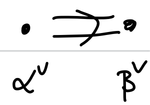
Type C_2

$$W = \underset{\langle s_\beta\rangle \times \langle s_{\beta+2\alpha}\rangle}{\underset{\shortparallel}{(C_2)^2}} \rtimes \underset{\langle s_\alpha\rangle}{\underset{\shortparallel}{S_2}}$$

$\Phi^\vee \subset Y(T)$



Type B_2



Observations: (a) $X(Z(G)) = X(T)/\mathbb{Z}\Phi = \mathbb{Z}/2$

So $Z(G) = \mu_2 \rightsquigarrow$ $\left[\begin{array}{cc|cc} -1 & & & \\ & -1 & & \\ \hline & & -1 & \\ & & & -1 \end{array}\right]$

(b) There is a <u>root string</u> of the form

$$\beta, \ \beta+\alpha, \ \beta+2\alpha$$

Observe $[\mathfrak{g}_{\alpha_1}, \mathfrak{g}_{\alpha_2}] = \mathfrak{g}_{\alpha_1+\alpha_2}$

Let $W = \bigoplus_n \mathfrak{g}_{\beta+n\alpha}$ $\rightsquigarrow$ W is a Z_α rep. $\rightsquigarrow$ SL_2 rep.

$$\mathrm{Ad}\left(\begin{array}{c|c} A & \\ \hline & A^{-t} \end{array}\right)\left(\begin{array}{c|c} & B \\ \hline & \end{array}\right) = \left(\begin{array}{c|c} & ABA^{t} \\ \hline & \end{array}\right)$$

So W is the three dim. irred. SL_2-rep. $S^2(k^2)$

Then length of the root string $=$ dim $W = |\underbrace{\langle \beta, \alpha^\vee \rangle}_{\text{lowest weight}}| + 1$

(c) $[Z_\beta, Z_{\beta+2\alpha}] = 1$, since $\beta+2\alpha \pm \beta \notin \Phi$

<u>Exercise</u>: Do SO_4 in the same detail. Also: Sp_{2n}!

2.10 Interlude: Rep. th. of SL_2

Denote by $V = k^2$, the std. rep. of SL_2.

Write $\quad sl_2 = \left\langle E = \begin{pmatrix} & 1 \\ & \end{pmatrix}, H = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, F = \begin{pmatrix} & \\ 1 & \end{pmatrix} \right\rangle$

So $[E,F] = H$, $[H,E] = 2E$, $[H,F] = -2F$.

For $n \geq 0$, write $L(\lambda) = S^\lambda(V) = \{ f \in k[x,y] \mid f \text{ hom. of degree } \lambda \}$

$$L(\lambda): \begin{tikzcd} \overset{-\lambda}{\bullet} \arrow[loop above] \arrow[r, shift left, "1"] & \overset{-\lambda+2}{\bullet} \arrow[loop above] \arrow[l, shift left, "\lambda"] \arrow[r, shift left, "2"] & \overset{-\lambda+4}{\bullet} \arrow[loop above] \arrow[l, shift left, "\lambda-1"] & \cdots & \overset{\lambda-4}{\bullet} \arrow[loop above] \arrow[r, shift left, "\lambda-1"] & \overset{\lambda-2}{\bullet} \arrow[loop above] \arrow[l, shift left, "2"] \arrow[r, shift left, "\lambda"] & \overset{\lambda}{\bullet} \arrow[loop above] \arrow[l, shift left, "1"] \end{tikzcd}$$

$$\xleftarrow{F} \quad \overset{H}{\circlearrowleft} \quad \xrightarrow{E}$$

$\{ \text{f.d. } SL_2\text{-rep.} \} \overset{1:1}{\longleftrightarrow} \{ \text{f.d. } sl_2\text{-rep.} \} \quad \rightsquigarrow \text{semisimple}$

$$W = \bigoplus_{\substack{\lambda \in \mathbb{Z}_{\geq 0} \\ \| \\ X(T)/W \, = \, X(T)_+}} L(\lambda)^{n_\lambda}$$

2.11 Abstract Root Data

<u>Defn:</u> An abstract root datum is a tuple

$$(X, \Phi, Y, \Phi^\vee, (\ )^\vee, \langle\ ,\ \rangle)$$

such that:

(1) X and Y are f.g. free abelian groups,

$\langle\ ,\ \rangle : X \otimes Y \to \mathbb{Z}$ is a perfect pairing

(2) $\Phi \subset X$, $\Phi^\vee \subset Y$ are finite subsets and

$(\ )^\vee : \Phi \to \Phi^\vee$ a bijection

(3) $\langle \alpha, \alpha^\vee \rangle = 2$

(4) $r_\alpha : X \to X$, $x \mapsto x - \langle x, \alpha^\vee \rangle \alpha$

is an automorphism of the root datum

(5) $\alpha \in \Phi \Rightarrow 2\alpha \notin \Phi$.

<u>Def B</u>: An abstract root system is a tuple

$$(V, \Phi, \Phi^\vee, ( )^\vee)$$ where

(1) V is a real vector space

(2) $\Phi \subset V$, $\Phi^\vee \subset V^*$ are finite subsets and

$( )^\vee : \Phi \to \Phi^\vee$ a bijection

(3) For $\alpha, \beta \in \Phi$, $\langle \alpha, \beta^\vee \rangle \in \mathbb{Z}$ and $\langle \alpha, \alpha^\vee \rangle = 2$

(4) $r_\alpha : V \to V$, $v \mapsto v - \langle v, \alpha^\vee \rangle \alpha$ and
$r_{\alpha^\vee} : V^* \to V^*$, $\lambda \mapsto \lambda - \langle \alpha, \lambda \rangle \alpha^\vee$
permute Φ and $\Phi^\vee$.

(5) If α and $c\alpha \in R$, then $c = \pm 1$

Remark: (1) There is a duality for root data:

$$(X, \Phi, Y, \Phi^\vee) \longleftrightarrow (Y, \Phi, X, \Phi^\vee)$$

(2) $(X, \Phi, Y, \Phi^\vee) \longmapsto (V, \Phi, \Phi^\vee, (\ )^\vee)$

$$V = \langle \Phi \rangle_{\mathbb{Z}} \otimes \mathbb{R}$$

yields a root system. In particular, there is a notion of irreducible root system, bases, positive roots Dynkin diagram, Weyl group, ..

(3) Root systems are equivalenty defined as (E, Φ) where $E = (E, (\ ,\ ))$ is a Euclidean v.s.

Then $\alpha^\vee = 2\frac{(-,\alpha)}{(\alpha,\alpha)}$.

2.12. Root datum of a reductive group

<u>Thm</u>: Let $G \supset T$ be a reductive group with max. torus

Then $(X(T), \Phi, Y(T), \Phi^\vee)$ is an (abstract)

root datum. Moreover the Weyl group $W(G,T)$

and $W(R, \Phi)$ coincide

<u>Proof</u>: We have to show 5 Axioms

(1) – (3) Clear by definition.

(4) Let $\alpha \in \Phi$. Then we obtain $s_\alpha \in W(Z_\alpha, T) \subset W(G,T)$

and $\tau_\alpha : \chi \mapsto \chi - \langle \chi, \alpha^\vee \rangle \alpha^\vee$.

We know that s_α induces an isomorphism of the root datum.

Let $V = SL_2\, \mathfrak{g}_\beta = \mathfrak{sl}_2\, \mathfrak{g}_\beta \subset \bigoplus_{n\in\mathbb{Z}} \mathfrak{g}_{\beta+n\alpha}$.

Then this is an irreducible representation.

Consider
$$\begin{aligned} X(T) &\longrightarrow X(\alpha^\vee(\mathbb{G}_m)) = \mathbb{Z} \\ \chi &\longmapsto \chi|_{\alpha^\vee(\mathbb{G}_m)} = \langle \chi, \alpha^\vee \rangle. \end{aligned}$$

So $V_{\chi|\alpha(\mathbb{G}_m)} = V_{\langle \chi, \alpha^\vee\rangle}$ and

$$s_\alpha(V_{\langle\chi,\alpha^\vee\rangle}) = V_{-\langle\chi,\alpha^\vee\rangle}.$$

Hence $\langle s_\alpha(\chi), \alpha^\vee\rangle = -\langle \chi, \alpha\rangle$.

Moreover, if $\langle \chi, \alpha^\vee\rangle = 0$, we have $s_\alpha(\chi) = \chi = \sigma_\alpha(\chi)$

Hence $s_\alpha = \sigma_\alpha$.

Moreover, we have

$$\begin{tikzcd}
W(\mathfrak{g}, T) \arrow[r, "\sim"] \arrow[d, "\text{torsor}"'] & W(R, \Phi) \arrow[d, "\text{torsor}"] \\
\mathcal{B}^T \arrow[u, Rightarrow] \arrow[r, leftrightarrow, "1:1"] & \text{Weyl chambers in } Y(T)_{reg}
\end{tikzcd}$$

(5) Exercise (Use sl_2-rep. th.)

Remark:

We obtain the following chain of W-torsors

$$
\begin{array}{ccc}
\mathcal{B}^T & & B(\lambda) \\
\uparrow & & \uparrow \\
\text{Weyl chambers in } Y(T)_{reg} & & C(\lambda) \\
\downarrow & & \downarrow \\
\text{Connected components of } Y(T)_{\mathbb{R}} - \bigcup_{\alpha\in\Phi} H_\alpha & & \text{“}C(\lambda)_{\mathbb{R}}\text{”} \\
\downarrow & & \downarrow \\
\text{Choice of positive roots} & & \Phi^+ = \{\alpha\in\Phi \mid \langle\alpha,\lambda\rangle > 0\} \\
\downarrow & & \downarrow \\
\text{Choice of simple roots} & & \Delta = \{\alpha\in\Phi^+ \mid \alpha\notin \mathbb{Z}_{\geq 0}\Phi^+\setminus\{\alpha\}\} \\
\downarrow & & \downarrow \\
\text{Choice of simple reflections} & & S = \{s_\alpha \mid \alpha\in\Delta\} \subset W
\end{array}
$$

In particular $\mathfrak{b} = \mathfrak{t} \oplus \bigoplus_{\alpha\in\Phi^+} \mathfrak{g}_\alpha$. $\square$

2.13 Weyl group and Type

Recall that the Dynkin diagram Γ arises from Δ by

Vertices $= \Delta$

$\#$ edges from α to $\beta = d_{\alpha,\beta} = \langle \alpha, \beta^\vee \rangle \langle \beta^\vee, \alpha \rangle \in \{0,1,2,3\}$
for $\alpha \neq \beta$

$$0 \to A_1 \times A_1, \quad 1 \to A_1, \quad 2 \to B_2, C_2, \quad 3 \to G_2$$

For $\langle \alpha, \beta^\vee \rangle \langle \beta^\vee, \alpha \rangle > 1$ and if $\langle \alpha, \beta^\vee \rangle = -1$ put an arrow

$$\underset{\alpha}{\bullet} \Leftarrow \underset{\beta}{\bullet}$$

The Weyl group is a Coxeter group with presentation

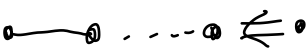
$$W = \langle s_\alpha \in S \mid s_\alpha^2 = 1, \ \underbrace{s_\alpha s_\beta \cdots}_{m_{\alpha,\beta}} = \underbrace{s_\beta s_\alpha \cdots}_{m_{\alpha\beta}}$$

where

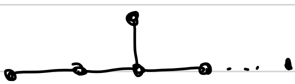
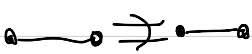
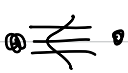
$m_{\alpha\beta}$	$d_{\alpha,\beta}$
2	0
3	1
4	2
6	3

For an irreducible root datum, the <u>type</u> is the type of the associated root system.

Classical

	A_n	$GL_{n+1}, SL_{n+1}, PGL_{n+1}$ (dual: $SL_{n+1} \leftrightarrow PGL_{n+1}$)
	B_n	SO_{2n+1} (dual: $SO_{2n+1} \leftrightarrow Sp_{2n}$)
	C_n	Sp_{2n}
	D_n	SO_{2n}

Exceptional

	E_6, E_7, E_8
	F_4
	G_2

Sketch:

$$\begin{tikzcd}
\{\text{root data}\}/\sim \arrow[r] \arrow[d, leftrightarrow, "1:1", "\text{To show}"'] & \{\text{root systems}\}/\sim \arrow[d, leftrightarrow] \\
\{\text{reductive groups over } k\}/\sim \arrow[r] & \{\text{semisimple Lie alg.}\}/\sim
\end{tikzcd}$$

$$G \longmapsto \mathrm{Lie}([G,G]) = \mathrm{Lie}(G/Z(G))$$

$\uparrow$

forgets center + isogeny (finite covers)

GL_2 vs. SL_2 $\qquad$ PGL_2 vs. SL_2

2.15 Generators, Automorphisms and simple constituents.

<u>Lemma</u>: Let $G \supset T$ be a reductive group with max. torus. Let $\Delta \subset \Phi$ be a set of simple roots. Then G is generated by $U_{\pm\alpha}, \alpha \in \Delta$, and T.

<u>Proof</u>: For $\alpha \in \Delta$, $Z_\alpha = C_G(\ker \alpha)^\circ)$ is generated by $U_{\pm\alpha}$ and T.

Let $H = \langle Z_\alpha \mid \alpha \in \Delta \rangle$. Then H contains $N_G(T)$, since W is generates by $s_\alpha \in N_{Z_\alpha}(T)/T$. Hence $\mathfrak{g}_{\sigma(\alpha)} \subset \mathfrak{h}$ for all $\alpha \in \Delta$, $\sigma \in W$. All elements in Φ arise this way. So $\mathfrak{h} = \mathfrak{g}$ which forces $H = G$ $\square$

Prop: Let $G \supset B \supset T$ be a reductive group with Borel and maximal torus. Let $\Delta \subset \Phi$ be the sets of simple roots. There is a natural morphism φ,

$$\begin{array}{l} D = \{\sigma \in \mathrm{Aut}(G) \mid \sigma(T) = T, \sigma(B) = B\}, \\ \downarrow \\ \Gamma = \{\sigma \in \mathrm{Aut}(X(T), \Phi) \mid \sigma(\Delta) = \Delta\} \end{array}$$

$(\varphi(\sigma))(\alpha) = \beta \iff \sigma(\mathfrak{g}_\alpha) = \mathfrak{g}_\beta$. Then:

(a) $\mathrm{Aut}(G) = \mathrm{Int}(G) D$ and

(b) $\varphi : \mathrm{Aut}/\mathrm{Int}(G) \hookrightarrow \Gamma$ is injective.

Proof: (a) Let $\sigma \in \mathrm{Aut}(\mathcal{G})$. Then there is $x \in \mathcal{G}$, s.t.

$x\,\sigma(B)x^{-1} = B$, and $y \in B$, s.t. $y x \sigma(T) x^{-1} y^{-1} = T$.

Hence $\mathrm{Int}(xy)\,\sigma \in D$.

(b) Let $\sigma \in D$, s.t. $\varphi(\sigma) = e \in \Gamma$.

Since $\varphi(\sigma) = e : X(T) \to X(T)$, $\sigma|_T = e$. Moreover

$\sigma(U_\alpha) = U_\alpha$ for all $\alpha \in \Delta$. Hence, we obtain

$$\mathrm{Aut}(U_\alpha) \longrightarrow \mathbb{G}_m,\ \varphi|_{U_\alpha} \longmapsto c_\alpha.$$

Since Δ is linearly independent, we find $t \in T$, s.t.

$\alpha(t) = c_\alpha$. Let $\sigma' = \mathrm{Int}(t)\,\sigma$. Then $\sigma'|_{U_\alpha} = e$ $\forall \alpha \in \Delta$

and $\sigma'|_T = e$. By the previous lemma, $\sigma' = e$. $\square$

<u>Thm</u>: Let $\mathcal{G}$ be semisimple and $\mathcal{S}$ the collection of minimal connected normal subgroups of positive dimension. Then

(a) $[H, H'] = 1$ for all $H \neq H' \in \mathcal{S}$.

(b) $|\mathcal{S}| = n < \infty$ and $\pi : H_1 \times \cdots \times H_n \to \mathcal{G}$ is surjective with finite kernel for all enumerations $\mathcal{S} = \{H_1, .., H_n\}$

(c) Let $\mathcal{G}' \subset \mathcal{G}$ be any connected subgroup of pos. dim. Then $\mathcal{G}'$ is the product of all groups in $\mathcal{S}_{\mathcal{G}'} = \{H \in \mathcal{S} \mid H \subset \mathcal{G}\}$, and $H \in C_{\mathcal{G}}(\mathcal{G}')$ $\forall H \in \mathcal{S} \setminus \mathcal{S}_{\mathcal{G}'}$.

(d) $\mathcal{G} = (\mathcal{G}, \mathcal{G})$

(e) $\mathcal{G}$ is generated by $U_{\pm\alpha}$, $\alpha \in \Delta$.

Proof: (1) $[H,H]$ is connected normal, so trivial by minimality.

(2) Let $\{H_1,\dots,H_m\} = \mathcal{S}' \subset \mathcal{S}$ be a finite subset.

By (a) $\pi: H_1 \times \dots \times H_m \to \mathfrak{g}$ is a hom. of groups.

and $H = \operatorname{im}\pi \trianglelefteq \mathfrak{g}$. We have $R(\mathfrak{g}') \subseteq R(\mathfrak{g}) = \{e\}$, so

H is semisimple. By (a) $H' \subset C_{\mathfrak{g}}(\mathfrak{g}')$ for all $H' \in \mathcal{S} \setminus \mathcal{S}'$.

and hence $H' \cap H$ is finite by minimality of H'.

Similarly $\ker \pi$ is finite by minimality. Hence $\mathcal{S}$

is finite by dimension. Let $\mathcal{S}' = \mathcal{S} = \{H_1, \dots, H_n\}$.

We show that $H = \mathfrak{g}$.

We obtain

$$\begin{array}{ccccccc} 1 \to & C_{\mathcal{G}}(H) & \to & \mathcal{G} & \longrightarrow & \mathrm{Aut}(H) \\ & \downarrow & & \downarrow & & \downarrow \\ 1 \to & C_{\mathcal{G}}(H)H/H & \to & \mathcal{G}/H & \longrightarrow & \mathrm{Aut}(H)/\mathrm{Int}(H) \end{array}$$

$\uparrow$ finite (Exercise)

Let $H' = (C_{\mathcal{G}}(H))^\circ$. So $H'H \subset \mathcal{G}$ has finite index.

By connectedness $H'H = \mathcal{G}$. Hence H' is normal in $\mathcal{G}$

which forces $H' = \{e\}$.

(c) similar to (a).

(d) $[H, H] \trianglelefteq \mathcal{G}$ for all $H \in \mathcal{S}$, so $[H, H] = H$ since

H minimal and $\mathcal{G}$ semisimple. So

$$[\mathcal{G}, \mathcal{G}] = \prod_{H \in \mathcal{S}} [H, H] = \prod_{H \in \mathcal{S}} H = \mathcal{G}.$$

(e) Exercise $\square$

<u>Corollary</u>: (a) The collection $\mathcal{S}$ from Theorem 1.14 is in bijection with the irreducible components of the root system.

(b) $\mathcal{B}_G = \prod_{H \in \mathcal{S}} \mathcal{B}_H$.

(b) For G reductive, $G = Z(G)\,[G,G]$.

<u>Proof</u>: Exercise. □

<u>Def</u>: A semisimple group G with irred. root system is called <u>almost simple</u>. □

Hence, if we want to understand the geometry of flag varieties, it suffices to consider almost simple groups

2.16 Reminder: Coxeter groups

Bourbaki, Lie groups and Lie algebras, Chapters 4–6

<u>Def:</u> Let W be a group generated by a set $S = S^{-1}$, $1 \notin S$

(1) The length $\ell(w) = \ell_S(w)$ is the smallest integer, such that w is a product of $\ell(w)$ elements in S.

(2) A sequence $\underline{w} = (s_1, \dots, s_n) \in S^n$ is called a <u>reduced expression</u> of w, if $s_1 \cdots s_n = w$ and $n = \ell(w)$ o

(3) The Bruhat order $\leq$ on W is defined by

$x \leq y \iff$ a reduced expression of x is a subsequence of a reduced expression of y. $\square$

<u>Thm</u> (Exchange condition) W has a presentation of the form $W = \langle s \in S \mid s^2 = 1,\ st^{m(s,t)} = 1 \rangle$ for suitable $m(s,t) \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$, i.e. is a Coxeter group, if and only if for each $w \in W$ with reduced expression $(s_1, \dots, s_n)$ and $s \in S$ with $\ell(sw) \leq \ell(w)$ there is a $1 \leq j \leq n$, s.t.,

$$s\, s_1 \cdots s_{j-1} = s_1 \cdots s_j\, s_{j+1}$$

Exchange!

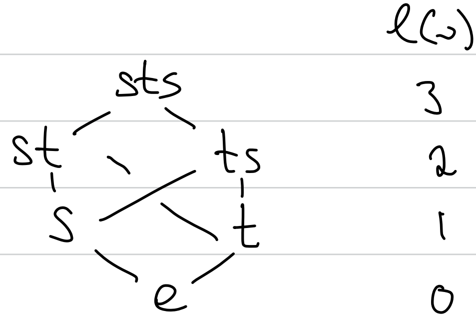
Corollary: Let $W \supset S$ be a Coxeter group, $w \in W$ with reduced expression $(s_1, \dots, s_n)$ and $s \in S$. Then, there are two cases:

(1) $\ell(sw) = \ell(w) + 1$, and $(s, s_1, \dots, s_n)$ is a reduced expression of sw, or

(2) $\ell(sw) = \ell(w) - 1$, and there is a $1 \leq j \leq n$, s.t. $(s_1, \dots, \hat{s}_j, \dots, s_n)$ and $(s, s_1, \dots, \hat{s}_j, \dots, s_n)$ are reduced expressions of ws and w, respectively.

Example: Let $W = \langle \underset{(1,2)}{\overset{\shortparallel}{s}}, \underset{(2,3)}{\overset{\shortparallel}{t}} \mid s^2 = t^2 = 1,\ sts = tst \rangle \cong S_3$

Then the Bruhat order and length are given by



$\square$

Exercise: For $W = S_n = \langle \underset{(1,2)}{\overset{\shortparallel}{s_1}}, \dots, \underset{(n-1,n)}{\overset{\shortparallel}{s_{n-1}}} \rangle$, determine m_{st}.

Show that $\ell(w) = |\{ i<j \mid w(i) > w(j) \}|$. $\square$

Remark: By a Theorem of Matsumoto in a Coxeter group any two reduced expressions of the same element are linked via braid relations, $sts\cdots \to tst\cdots$

2.17 Tits system, Bruhat decomposition and Coxeter group

In this section, we deal with abstract groups.

<u>Definition</u>: A Tits system is a tuple

$$(\mathcal{G}, B, N, S)$$

where $\mathcal{G}$ is a group with subgroups B, N and

$S \subset W = N/_{B \cap N}$ is a subset, such that:

(1) $B \cup N$ generates $\mathcal{G}$, $B \cap N \trianglelefteq N$

(2) S generates $W = N/(B \cap N)$ and $|s| = 2 \ \forall s \in S$

(3) $sBw \subset BwB \cup BswB$ for $s \in S$, $w \in W$.

(4) $sBs \not\subset B$ for all $s \in S$. $\square$

Think: $\underset{\substack{\uparrow \\ \text{reductive}}}{(G}, \underset{\substack{\uparrow \\ \text{Borel}}}{B}, \underset{\substack{\uparrow \\ N_G(T)}}{N}, \underset{\substack{\uparrow \\ \text{simple reflections in } W \\ \text{associated to } B.}}{S)}$ $\square$

Fix a Tits system (G, B, N, S).

Proposition: Let $s \in S$, $w, w' \in W$ and $\ell = \ell_S : W \to \mathbb{Z}_{\geq 0}$ the length function. Then

(1) $Bww'B \subset BwBw'B$, $Bw^{-1}B = (BwB)^{-1}$.

(2) $BsBwB \subset BswB \cup BwB$

(3)
$$BsBwB = \begin{cases} BswB & \text{if } \ell(sw) = \ell(w)+1 \\ BwB \cup BswB & \text{if } \ell(sw) = \ell(w)-1 \end{cases}$$

(4) $BwBsB = BwBwBwsB, \ldots$

(5) For $s_1, \ldots, s_n \in S$, we get

$$Bs_1 \cdots s_n BwB \subset \bigcup_{1 \leq i_1 < \cdots < i_m \leq p} B s_{i_1} \cdots s_{i_m} wB$$

Proof: Exercise $\square$

Thm A (a) For $I \subset S$, let $W_I = \langle s \in I \rangle$, $P_I = BW_IB$.

Then $P_I \subset G$ is a subgroup, called parabolic subgroup

(b) $G = BWB = \bigoplus_{w \in W} BwB$. This is called the

Bruhat decomposition of G.

Pf: (a) Use previous Proposition (5)

(b) By (a) $BWB \subset G$ is a subgroup. Since $N, B \subset BWB$

generate G, $BWB = G$. It remains to show

$w \neq w' \Rightarrow BwB \neq Bw'B$. Let $\ell = \ell_S : W \to \mathbb{Z}_{\geq 0}$ the length.

W.l.o.g. $\ell(w) \geq \ell(w') = n$. We show the statement by

induction on n.

If $q=0$, $w'=e$ and $Bw'B=B\neq BwB$.

If $q\geq 1$, choose $s\in S$, s.t., $\ell(sw')=q-1$.

Hence $\ell(w)>\ell(w's)$ and $w\neq sw'$ and $sw\neq sw'$

and $\ell(sw)\geq \ell(w)-1\geq \ell(sw')=q-1$.

By induction, $Bsw'B\neq BwB, BswB$, so

$$Bsw'B\cap BsBBwB=Bsw'B\cap(BwB\cup BswB)=\emptyset$$

which shows

$$Bw'B\cap BwB\subset BsBBsw'B\cap BwB=\emptyset \qquad \square$$

Thm B: $W \supset S$ is a Coxeter group.

Proof: We show that $W \supset S$ fulfill the exchange condition (see 2.16 Theorem). Let $w \in W$ with reduced expression $(s_1, \ldots, s_n)$ and $s \in S$, s.t., $\ell(sw) \leq \ell(w)$.

By Prop. (3), $BwB \subset BsBwB$, so

$$BsB \subset BwBw^{-1}B \subset \biguplus_{1 \leq i_1 < \ldots < i_m \leq n} Bw\, s_{i_m} \cdots s_{i_1} B$$

So $s = w s_{i_m} \cdots s_{i_1}$ for a unique choice $1 \leq i_1 < \ldots < i_m \leq n$

and $w = s s_{i_1} \cdots s_{i_m}$, so $m = n-1$. Let $1 \leq j \leq n$, s.t., $\{i_1, \ldots, i_n\} \cup \{j\} = \{1, \ldots, n\}$. Then

$$s s_1 \cdots s_{j-1} = s_1 \cdots s_j$$

$\square$

2.18. Parabolic subgroups and Simplicity

As in the last Section, let $(\mathcal{G}, B, N, S)$ be a Tits system.

<u>Lemma A</u>: (1) Let $w \in W$ with reduced expression $(s_1, \ldots, s_n)$, let $I = \{s_1, \ldots, s_n\}$, then

$$P_I = BW_IB = \langle B, wBw^{-1} \rangle = \langle B, w \rangle$$

(2) $S = \{w \in W \mid B \cup BwB \subset \mathcal{G} \text{ is a subgroup}\}$ and S is a minimal generating set of W

(3) Let $w \in W$, $I, J \subset S$. If $wP_I w^{-1} \subset P_J$, then $w \in P_J$.

<u>Proof</u>: Exercise

Thm A: (a) The map

$$2^S \longrightarrow \{B \subset P \subset G\} \quad I \longmapsto P_I$$

is a bijection.

(b) If P_J is conjugate to P_I, $P_I = P_J$

(c) $N_G(P_I) = P_I$

Pf: (a) Surj: Let $B \subset P \subset G$. Then P is a union of double cosets BwB. Let I consist of all $s \in S$ that appear in reduced expressions for $w \in P$. Then clearly $P \subset P_I$. Lemma (1) implies $P_I \subset P$.

Inj: Follows from lemma (2)

(b) Since B, N generate $\mathcal{G}$, P_I is conjugate to P_J, iff

$w P_I w^{-1} = P_J$ for some $w \in W$. Then use lemma 3

(c) Same as (b) $\square$

Lemma B: Let $H \trianglelefteq \mathcal{G}$. Then there is a partition $S = I \uplus J$

such that $[I, J] = \{e\}$ and $HB = P_I$.

Proof: Since $H \trianglelefteq \mathcal{G}$, HB is a subgroup.

By Thm. A(a), $HB = P_I$ for some $I \subset S$.

Let $J = S \setminus I$. We have $I = \{s \in S \mid BsB \cap H \neq \emptyset\}$

(Easy Exercise). We need to show $[I, J] = \{e\}$.

Let $s \in I$, $t \in J$. Then $\ell(st) \geq 2$, so

$sBt \subset BstB$ and $tBsBt \subset BtsB \cup BtstB$

We have $H \cap BsB \neq \emptyset$. Since $H \trianglelefteq G$, we get

$H \cap tBsBt \neq \emptyset$. Hence $\{ts, tst\} \cap W_I \neq \emptyset$.

Now $ts \notin W_I$ since $t \notin W_I$. So

$$tst \in W_I \cap W_{\{s,t\}} = W_{\{s\}} = \{1, s\}.$$

Hence $tst = s \Rightarrow ts = st$ $\square$

We say W is irreducible if there is no partition

$I \uplus J = S$, s.t., $W = W_I \times W_J$

Thm B: Let W be irreducible and assume

(1) G is generated by conjugates of a solvable $U \trianglelefteq B$

(2) $G = [G, G]$

Then G/Z is simple (or trivial).

Proof: Let $H \trianglelefteq G$, $H \not\leq Z$. By Lemma B, and irreducibility of W, $G = HB$. Using (1), $G = HU$.

Then $U/_{U \cap H} \cong HU/_H \cong G/H$.

Now the left side is solvable and the right side perfect. Hence both are trivial. $\square$

Outlook:

BN-pair:

$\rightsquigarrow$ $(G, B, N_G(T), S)$ is a BN-pair

$\rightsquigarrow$ G/Z simple for G with indec. root datum

$\rightsquigarrow$ $U_w \times \{\dot{w}\} \times B \xrightarrow{\sim} B\dot{w}B$ $\quad U_w = U \cap \dot{w}U^{-}\dot{w}^{-1}$

$\rightsquigarrow$ $G/B = \biguplus \underbrace{B\dot{w}B/B}_{\cong U_w \cong \mathbb{A}^{\ell(w)}}$

Rep-th.:

$\lambda \in X(T) \rightsquigarrow \mathcal{L}_\lambda$ line bundle on G/B

$\rightsquigarrow$ $H^0(G/B, \mathcal{L}_\lambda) = L(\lambda)$

$\rightsquigarrow$ $\mathcal{O}(G) = \bigoplus L(\lambda) \otimes L(\lambda)^*$

$\rightsquigarrow$ Weyl's character formula = excess intersection formula

Summary

Zariski topology

$$X = V(I = \sqrt{I}) \subset k^n \qquad \{\text{affine variety}\}$$

$$\updownarrow \qquad\qquad\qquad\qquad \updownarrow$$

$$\mathcal{O}(X) = k[x_1, \dots, x_n]/I \qquad \{\text{reduced f.g. } k\text{-algebra}\}^{op}$$

glue!

$$X = (X, \mathcal{O}_X) \quad \text{top. space + sheaf of functions}$$

Example: $X = \mathbb{P}^n$, $Gr(k,n)$, $V(I) \subset \mathbb{P}^n$

hom. ideal

$$T_x X \rightsquigarrow \text{derivations}, \; k[\varepsilon]/\varepsilon^2, \; (\mathfrak{m}_{X,x}/\mathfrak{m}_{X,x}^2)^*, \dots$$

G affine algebraic group $\Leftrightarrow$ $\mathcal{O}(G)$ comm. red. f.g. Hopf algebra

$\Updownarrow$

$G \subset GL_n$ linear algebraic group

$G \rightsquigarrow \mathcal{O}(G)$

Quotients

$$H \subset G \quad \rightsquigarrow \quad \begin{array}{ccc} I(H) & \subset & \mathcal{O}(G) \\ \cup & & \cup \\ U & \subset & W \end{array} \quad \text{f.d.}$$

$$H = G_{I(H)}$$

$$H = G_U$$

$$\Downarrow$$

$$L = \Lambda^d U \subset \Lambda^d W \qquad H = G_L$$

$$\Downarrow$$

$$x = [L] \subset \mathbb{P}(V) \qquad H = G_x$$

$$\Downarrow$$

$$G/H \cong Gx \quad \text{quasi projective}$$

Lie algebras

$$G \rightsquigarrow \mathfrak{g} = \mathrm{Lie}(G) = T_e X = \mathrm{Der}_g(\mathcal{O}(G))$$

$$GL_n \rightsquigarrow \mathfrak{gl}_n$$

$$\mathrm{Ad} : G \rightsquigarrow \mathfrak{g}, \qquad \mathrm{ad} : \mathfrak{g} \rightsquigarrow \mathfrak{g}$$

"gYg^{-1}" "$[X,Y]$"

Anatomy of a group element

$$X = X_s + X_n, \qquad x = x_s x_u = x_u x_s$$

$\rightsquigarrow$ Intrinsical!

Anatomy of a group

$$\{1\} \;\text{---}\; R_u(G^\circ) \;\text{---}\; R(G^\circ) \;\text{---}\; G^\circ \;\text{---}\; G$$

connected

finite

solvable
$\left(\begin{smallmatrix} \ast & \ast \\ & \ast \end{smallmatrix}\right)$
semidirect product

semisimple = almost prod. of almost simple
$\rightsquigarrow SL_n, PGL_n, \dots$

unipotent
$\rightsquigarrow G_a, \left(\begin{smallmatrix} 1 & \ast \\ & \ddots & \\ & & 1 \end{smallmatrix}\right)$

torus
$\rightsquigarrow G_m^n$

reductive
$\rightsquigarrow GL_n$

Diagonalizable Groups

$H \subset \mathbb{G}_m^n \iff H = H_{ss}$

$X(H)$, $Y(H)$, $\langle , \rangle$

$\{\text{diag. groups}\} \overset{\sim}{\longleftrightarrow} \{\text{f.g. Abelian groups}\}^{op}$

$H \longmapsto X(H)$

$\mathrm{Spec}(k[M]) \longleftarrow\!\!\shortmid M$

Flag variety

torus Borel reductive

$$\underbrace{T \subset B \subset G}$$

G/B projective

$$\cong$$

$\mathcal{B} = \{\text{Borel subgroups}\} \cong \{\text{Borel subalgebras}\}$

$\parallel$

"flag variety"

$$W = W(G,T) \curvearrowright \mathcal{B}^T$$

$\uparrow$
torsor

G s.s. rank 1

$$\begin{tikzcd}
SL_2 \arrow[r] \arrow[dr] & G \arrow[r] \arrow[d] & PGL_2 = \mathrm{Aut}(\mathbb{P}^1) \arrow[dl] \\
 & \mathcal{B} = \mathbb{P}^1 &
\end{tikzcd}$$

Weights and Roots

$$\Phi_{red} \underset{\text{if } G \text{ reductive}}{\overset{=}{\subset}} \Phi \subset X(T)$$

$$\mathrm{Lie}(G/R(G))$$

$$R(G) = \bigcap_{B \in \mathcal{B}} B$$

Root Subgroups

G reductive $\quad \alpha \in X(T)$

$\rightsquigarrow T_\alpha \subset Z_\alpha \leftarrow SL_2$

$$U_\alpha \leftarrow \begin{pmatrix} 1 & * \\ & 1 \end{pmatrix}$$

$$T \supset T_\alpha \longleftarrow \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} \longleftarrow\!\shortmid\, t \qquad \alpha^\vee$$

coroot

Root Datum

$(X(T), \Phi, Y(T), \Phi^\vee)$ $\Longleftrightarrow$ $T \subset G$

root datum $\qquad$ reductive group

$\downarrow$ $\qquad$ $\downarrow$

$(X(T)_{\mathbb{R}}, \Phi)$ $\qquad$ $\mathfrak{t} \subset \mathfrak{g}$

root system $\qquad$ Lie alg.

$\|$

$A\, B\, C \ldots G$

W-torsors

$T \subset B \subset G$

$\Phi^+ \subset \Phi$

$\Delta \subset \Phi$

$S \subset W$

Tits System

$\mathcal{G}$ reductive

$\Rightarrow (\mathcal{G}, B, N, S)$ Tits system

$\mathcal{G} = BWB, \quad \{B \subset P_I \subset \mathcal{G}\} \Leftrightarrow \{I \subset S\}$