Perverse Sheaves
on
Flag Varieties

Jens Niklas Eberhardt

I. Perverse sheaves

1. Notation

$$X = \bigsqcup_{\lambda \in \Lambda} X_\lambda$$

1. Notation

$$\overset{\text{variety }/\mathbb{C}}{\overset{\downarrow}{X}} = \bigsqcup_{\lambda \in \Lambda} X_\lambda$$

1. Notation

$$\overset{\substack{\text{variety }/\mathbb{C}\\ \downarrow}}{X} = \overset{\substack{\text{"nice" stratification}\\ \downarrow}}{\biguplus_{\lambda\in\Lambda}} X_\lambda$$

1. Notation

$$\overset{\substack{\text{variety } /\mathbb{C}\\ \downarrow}}{X} = \overset{\substack{\text{"nice" stratification}\\ \downarrow}}{\biguplus_{\lambda\in\Lambda}} X_\lambda \longleftarrow \text{smooth + connected}$$

1. Notation

$$\overset{\substack{\text{variety } /\mathbb{C} \\ \downarrow}}{X} = \overset{\substack{\text{"nice" stratification} \\ \downarrow}}{\bigsqcup_{\lambda\in\Lambda}} X_\lambda \quad \begin{array}{l} \leftarrow \text{smooth + connected} \\ \leftarrow d_\lambda = \dim_{\mathbb{C}} X_\lambda \end{array}$$

1. Notation

$$\overset{\text{variety } /\mathbb{C}}{\overset{\downarrow}{X}} = \overset{\text{"nice" stratification}}{\overset{\downarrow}{\bigsqcup_{\lambda\in\Lambda}}} X_\lambda \Leftarrow \begin{cases} \text{smooth + connected} \\ d_\lambda = \dim_{\mathbb{C}} X_\lambda \\ i_\lambda : X_\lambda \hookrightarrow X \quad \text{affine} \end{cases}$$

1. Notation

$$\overset{\text{variety }/\mathbb{C}}{\downarrow}\qquad\qquad \overset{\text{"nice" stratification}}{\downarrow}$$

$$X = \bigsqcup_{\lambda\in\Lambda} X_\lambda$$

smooth + connected

$d_\lambda = \dim_{\mathbb{C}} X_\lambda$

$i_\lambda : X_\lambda \hookrightarrow X$ affine

Example $\mathbb{P}^1_{\mathbb{C}}$

1. Notation

$$\overset{\text{variety }/\mathbb{C}}{X} = \overset{\text{"nice" stratification}}{\bigsqcup_{\lambda \in \Lambda}} X_\lambda$$

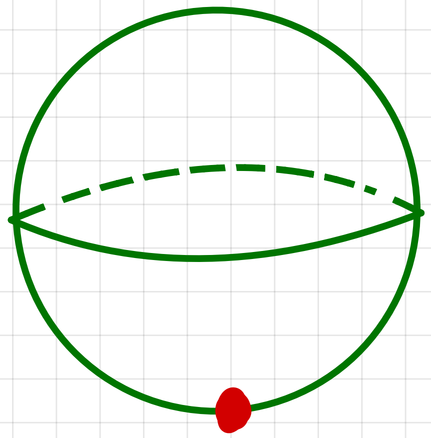
smooth + connected

$d_\lambda = \dim_{\mathbb{C}} X_\lambda$

$i_\lambda : X_\lambda \hookrightarrow X$ affine

Example

$$\mathbb{P}^1_{\mathbb{C}} = pt \sqcup \mathbb{C}$$



2. Local systems

$$\mathrm{Loc}(X_\lambda) \subset \mathrm{Sh}(X_\lambda, k)$$

2. Local systems

$$\mathrm{Loc}(X_\lambda) \subset \overset{\substack{\text{sheaves}\\ \downarrow}}{\mathrm{Sh}}(X_\lambda, \overset{\substack{\text{coeff. field}\\ \downarrow}}{k})$$

2. Local systems

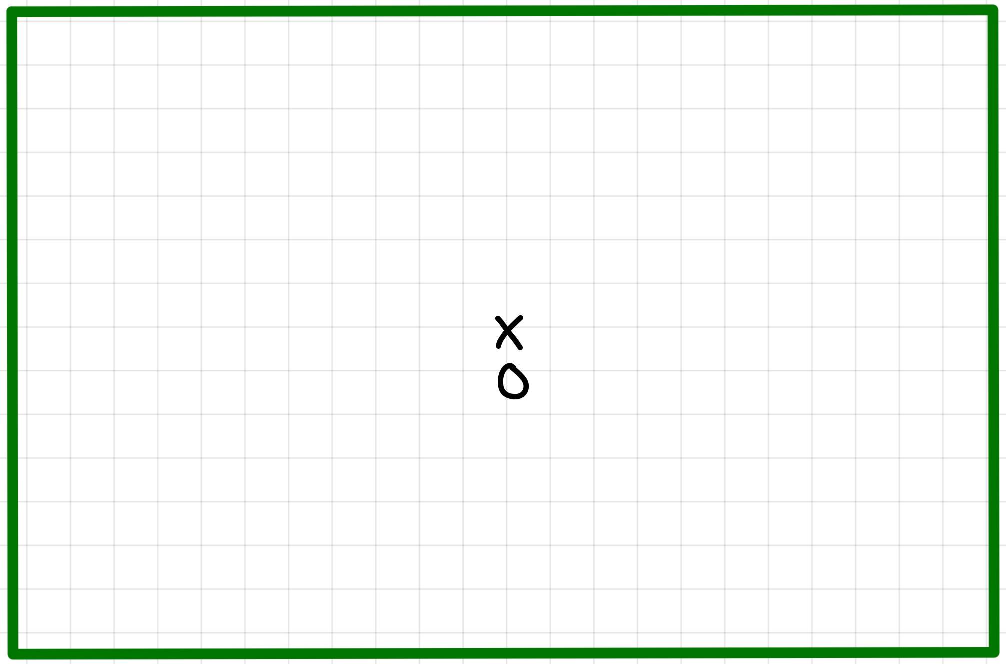
$$\overset{\text{local systems}}{\overset{\downarrow}{\mathrm{Loc}}}(X_\lambda) \subset \overset{\text{sheaves}}{\overset{\downarrow}{\mathrm{Sh}}}(X_\lambda, \overset{\text{coeff. field}}{\overset{\downarrow}{\Bbbk}})$$

2. Local systems

$$\begin{tikzcd}
& \text{local systems} \arrow[d] & & \overset{\text{sheaves}}{\downarrow} \qquad \overset{\text{coeff. field}}{\downarrow} \\
\mathcal{F} \arrow[d, mapsto] & \mathrm{Loc}(X_\lambda) \arrow[d] & \subset & \mathcal{S}h(X_\lambda, \Bbbk) \\
\mathcal{F}_x \arrow[r, phantom, "\in"] & \Bbbk\pi_1(X_\lambda, x)\text{-mod} & & \\
\text{stalk} \arrow[u] & & &
\end{tikzcd}$$

2. Local systems

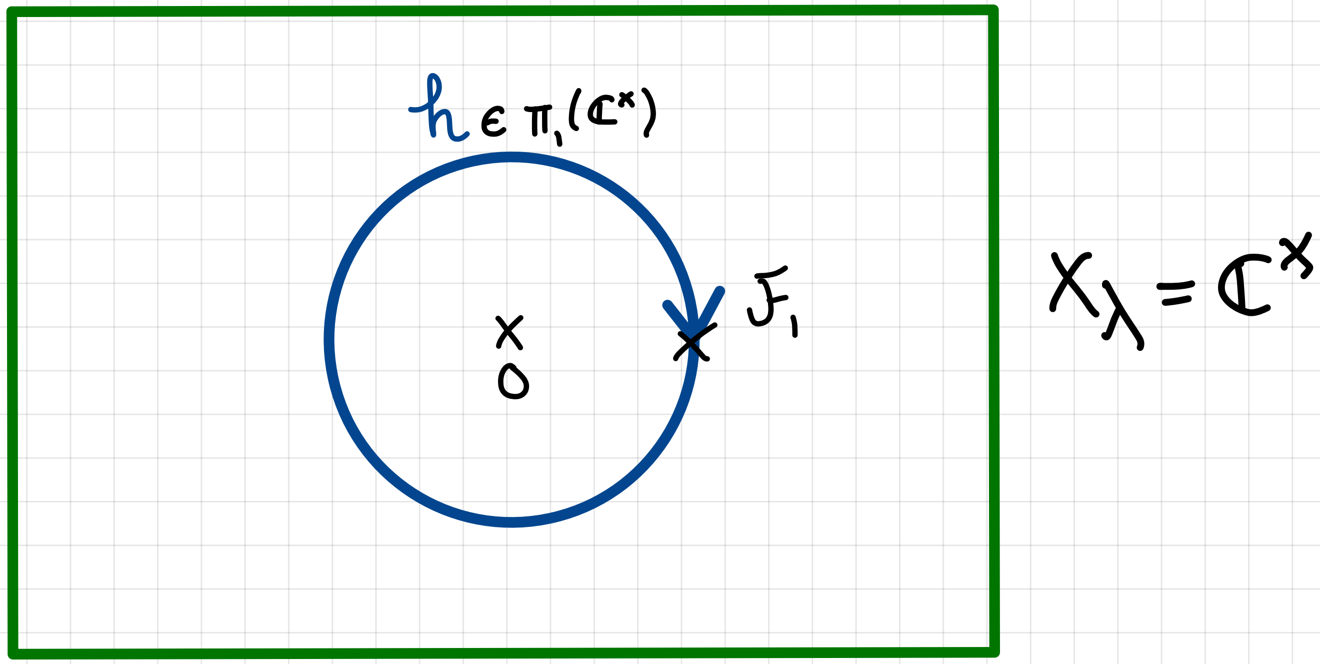
$$\begin{tikzcd}
\mathcal{F} \arrow[d, mapsto] & \overset{\substack{\text{local systems}\\ \downarrow}}{\mathrm{Loc}}(X_\lambda) \arrow[d] & \subset & \overset{\substack{\text{sheaves}\\ \downarrow}}{Sh}(X_\lambda, \overset{\substack{\text{coeff. field}\\ \downarrow}}{k}) \\
\underset{\substack{\uparrow\\ \text{stalk}}}{\mathcal{F}_x} & \in k\pi_1(X_\lambda, x)\text{-mod} & &
\end{tikzcd}$$



$X_\lambda = \mathbb{C}^\times$

2. Local systems

$$\begin{tikzcd}
\mathcal{F} \arrow[d, mapsto] & \overset{\text{local systems}}{\overset{\downarrow}{\mathrm{Loc}}}(X_\lambda) \arrow[d] \arrow[r, phantom, "\subset"] & \overset{\text{sheaves}}{\overset{\downarrow}{\mathrm{Sh}}}(X_\lambda, \overset{\text{coeff. field}}{\overset{\downarrow}{k}}) \\
\underset{\overset{\uparrow}{\text{stalk}}}{\mathcal{F}_x} \arrow[r, phantom, "\in"] & k\pi_1(X_\lambda, x)\text{-mod} &
\end{tikzcd}$$



3. Constructible sheaves

$\mathcal{H}^i(\mathcal{F}) \in Loc(X_\lambda)$ bounded derived category

$$D^b_{const}(X_\lambda) \subset D^b(Sh(X_\lambda))$$

3. Constructible sheaves

$\mathcal{H}^i(\mathcal{F}) \in Loc(X_\lambda)$ bounded derived category

$$\begin{tikzcd}
D^b_{const}(X_\lambda) \arrow[r, phantom, "\subset"] & D^b(Sh(X_\lambda)) \\
D^b_\Lambda(X) \arrow[u, "i_\lambda^*"] \arrow[r, phantom, "\subset"] & D^b(Sh(X)) \arrow[u, "i_\lambda^*"]
\end{tikzcd}$$

Λ-constructible category

$i_\lambda^* \mathcal{F} \in D^b_{const}(X_\lambda) \ \forall \lambda$

3. Constructible sheaves

$\mathcal{H}^i(\mathcal{F}) \in Loc(X_\lambda)$

bounded derived category

$$\begin{tikzcd}
D^b_{const}(\mathcal{X}_\lambda) \arrow[r, phantom, "\subset"] & D^b(Sh(\mathcal{X}_\lambda)) \\
D^b_\Lambda(\mathcal{X}) \arrow[u, "i_\lambda^*"'] \arrow[r, phantom, "\subset"] & D^b(Sh(\mathcal{X})) \arrow[u, "i_\lambda^*"']
\end{tikzcd}$$

Λ - constructible category

$i_\lambda^* \mathcal{F} \in D^b_{const}(X_\lambda) \ \forall \lambda$

4. Perverse sheaves

$$\mathrm{Loc}(X_\lambda)[d_\lambda] \subset D^b_{\mathrm{const}}(X_\lambda)$$

4. Perverse sheaves

$$\overset{\text{Verdier duality}}{\overset{\downarrow}{\mathbb{D}}} \overset{(*)}{\circlearrowright} \mathrm{Loc}(X_\lambda)[d_\lambda] \subset D^b_{const}(X_\lambda)$$

(*) $\mathcal{L} \in \mathrm{Loc}(X_\lambda)$

$$\mathbb{D}(\mathcal{L}[d_\lambda]) \underset{\substack{\uparrow \\ X_\lambda \text{ smooth}}}{=} \mathcal{L}^\vee[2d_\lambda - d_\lambda] = \mathcal{L}^\vee[d_\lambda]$$

4. Perverse sheaves

$$\overset{\substack{\text{Verdier duality}\\ \downarrow}}{\mathbb{D}} \overset{(*)}{\circlearrowright} \mathrm{Loc}(X_\lambda)[d_\lambda] = D^b_{\mathrm{const}}(X_\lambda)^{\overset{\substack{\text{std. t-structure shifted by } d_\lambda\\ \downarrow}}{t=0}}$$

(*) $\mathcal{L} \in \mathrm{Loc}(X_\lambda)$

$$\mathbb{D}(\mathcal{L}[d_\lambda]) \underset{\substack{\uparrow\\ X_\lambda \text{ smooth}}}{=} \mathcal{L}^\vee[2d_\lambda - d_\lambda] = \mathcal{L}^\vee[d_\lambda]$$

4. Perverse sheaves

Verdier duality

std. t-structure shifted by d_λ

$$\begin{tikzcd}
\mathbb{D} \overset{(*)}{\circlearrowright} \mathrm{Loc}(X_\lambda)[d_\lambda] \arrow[d, squiggly] & = \arrow[d, phantom, "\text{gluing}"] & D^b_{\mathrm{const}}(X_\lambda)^{t=0} \arrow[d, squiggly] \\
\mathbb{D} \circlearrowright \mathrm{Perv}_\Lambda(X) & = & D^b_\Lambda(X)^{t=0}
\end{tikzcd}$$

(*) $\mathcal{L} \in \mathrm{Loc}(X_\lambda)$

$$\mathbb{D}(\mathcal{L}[d_\lambda]) \underset{\substack{\uparrow \\ X_\lambda \text{ smooth}}}{=} \mathcal{L}^\vee[2d_\lambda - d_\lambda] = \mathcal{L}^\vee[d_\lambda]$$

4. Perverse sheaves

Verdier duality

std. t-structure shifted by d_λ

$$\begin{tikzcd}
\mathbb{D} \overset{(*)}{\circlearrowright} \mathrm{Loc}(X_\lambda)[d_\lambda] \arrow[d, squiggly] \arrow[rr, phantom, "="] & & D^b_{\mathrm{const}}(X_\lambda)^{t=0} \arrow[d, squiggly] \\
\mathbb{D} \circlearrowright \mathrm{Perv}_\Lambda(X) \arrow[rr, phantom, "="] & \text{gluing} & D^b_\Lambda(X)^{t=0}
\end{tikzcd}$$

perverse t-structure

$i_\lambda^! \mathcal{F} \in D_{\mathrm{const}}(X_\lambda)^{t \geq 0}$

$i_\lambda^* \mathcal{F} \in D_{\mathrm{const}}(X_\lambda)^{t \leq 0}$

(*) $\mathcal{L} \in \mathrm{Loc}(X_\lambda)$

$$\mathbb{D}(\mathcal{L}[d_\lambda]) \underset{\substack{\uparrow \\ X_\lambda \text{ smooth}}}{=} \mathcal{L}^\vee[2d_\lambda - d_\lambda] = \mathcal{L}^\vee[d_\lambda]$$

4. Perverse sheaves

Verdier duality

std. t-structure shifted by d_λ

$(*)$

$$\begin{tikzcd}
\mathbb{D} \circlearrowright \mathrm{Loc}(X_\lambda)[d_\lambda] \arrow[d, rightsquigarrow] & = & D^b_{\mathrm{const}}(X_\lambda)^{t=0} \arrow[d, rightsquigarrow] \\
 & \text{gluing} & \\
\mathbb{D} \circlearrowright \mathrm{Perv}_\Lambda(X) \arrow[d, rightsquigarrow] & = & D^b_\Lambda(X)^{t=0} \\
\mathcal{H}^{\neq 0}(\mathcal{F}) = 0 & &
\end{tikzcd}$$

perverse t-structure

$i_\lambda^! \mathcal{F} \in D_{\mathrm{const}}(X_\lambda)^{t \geq 0}$

$i_\lambda^* \mathcal{F} \in D_{\mathrm{const}}(X_\lambda)^{t \leq 0}$

$(*)$ $\mathcal{L} \in \mathrm{Loc}(X_\lambda)$

$\mathbb{D}(\mathcal{L}[d_\lambda]) \overset{\uparrow}{=} \mathcal{L}^\vee[2d_\lambda - d_\lambda] = \mathcal{L}^\vee[d_\lambda]$

X_λ smooth

4. Perverse sheaves

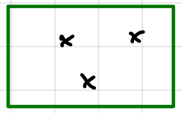
Verdier duality

std. t-structure shifted by d_λ

$$\begin{tikzcd} \mathbb{D} \overset{(*)}{\circlearrowright} \mathrm{Loc}(X_\lambda)[d_\lambda] \arrow[d, squiggly] & = & D^b_{const}(X_\lambda)^{t=0} \arrow[d, squiggly] \\ \mathbb{D} \circlearrowright \mathrm{Perv}_\Lambda(X) \arrow[d, squiggly] & = & D^b_\Lambda(X)^{t=0} \\ \mathcal{H}^{>0}(\mathcal{F}) = 0 & & \end{tikzcd}$$

gluing

$\mathcal{H}^0(\mathcal{F})$



perverse t-structure

$i_\lambda^! \mathcal{F} \in D_{const}(X_\lambda)^{t\geq 0}$

$i_\lambda^* \mathcal{F} \in D_{const}(X_\lambda)^{t\leq 0}$

(*) $\mathcal{L} \in \mathrm{Loc}(X_\lambda)$

$\mathbb{D}(\mathcal{L}[d_\lambda]) \underset{\uparrow}{=} \mathcal{L}^\vee[2d_\lambda - d_\lambda] = \mathcal{L}^\vee[d_\lambda]$

X_λ smooth

4. Perverse sheaves

Verdier duality

std. t-structure shifted by d_λ

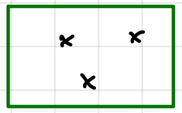
$$\begin{tikzcd}
\mathbb{D} \overset{(*)}{\circlearrowright} & \mathrm{Loc}(X_\lambda)[d_\lambda] \arrow[d, squiggly] \arrow[dr, phantom, "\text{gluing}"] & = & D^b_{const}(X_\lambda)^{t=0} \arrow[d, squiggly] \\
\mathbb{D} \circlearrowright & \mathrm{Perv}_\Lambda(X) \arrow[d, squiggly] & = & D^b_\Lambda(X)^{t=0} \\
 & \mathcal{H}^{>0}(\mathcal{F}) = 0 & &
\end{tikzcd}$$

perverse t-structure

$i_\lambda^! \mathcal{F} \in D_{const}(X_\lambda)^{t \geq 0}$

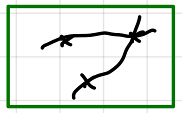
$i_\lambda^* \mathcal{F} \in D_{const}(X_\lambda)^{t \leq 0}$

$\mathcal{H}^{0}(\mathcal{F})$



$\vdots$

$\mathcal{H}^{-d_X+1}(\mathcal{F})$



(*) $\mathcal{L} \in \mathrm{Loc}(X_\lambda)$

$\mathbb{D}(\mathcal{L}[d_\lambda]) = \mathcal{L}^\vee[2d_\lambda - d_\lambda] = \mathcal{L}^\vee[d_\lambda]$

X_λ smooth

4. Perverse sheaves

Verdier duality

(*)

std. t-structure shifted by d_λ

$$\begin{tikzcd} \mathbb{D} \circlearrowright \mathrm{Loc}(X_\lambda)[d_\lambda] \arrow[r, phantom, "="] \arrow[d, rightsquigarrow] & D^b_{const}(X_\lambda)^{t=0} \arrow[d, rightsquigarrow] \\ \mathbb{D} \circlearrowright \mathrm{Perv}_\Lambda(X) \arrow[r, phantom, "="] \arrow[d, rightsquigarrow] & D^b_\Lambda(X)^{t=0} \\ \mathcal{H}^{>0}(\mathcal{F}) = 0 & \end{tikzcd}$$

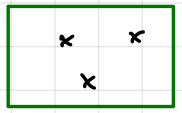
gluing

perverse t-structure

$i_\lambda^! \mathcal{F} \in D_{const}(X_\lambda)^{t\geq 0}$

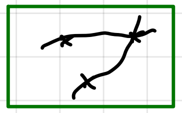
$i_\lambda^* \mathcal{F} \in D_{const}(X_\lambda)^{t\leq 0}$

$\mathcal{H}^{0}(\mathcal{F})$

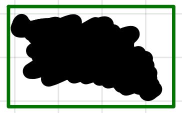


$\vdots$

$\mathcal{H}^{-d_X+1}(\mathcal{F})$



$\mathcal{H}^{-d_X}(\mathcal{F})$



(*) $\mathcal{L} \in \mathrm{Loc}(X_\lambda)$

$\mathbb{D}(\mathcal{L}[d_\lambda]) \underset{\uparrow}{=} \mathcal{L}^\vee[2d_\lambda - d_\lambda] = \mathcal{L}^\vee[d_\lambda]$

X_λ smooth

4. Perverse sheaves

Verdier duality

std. t-structure shifted by d_λ

$$\begin{tikzcd} \mathbb{D} \overset{(*)}{\circlearrowright} \mathrm{Loc}(X_\lambda)[d_\lambda] \arrow[d, squiggly] & = & D^b_{\mathrm{const}}(X_\lambda)^{t=0} \arrow[d, squiggly] \\ \mathbb{D} \circlearrowright \mathrm{Perv}_\Lambda(X) \arrow[d, squiggly] & = & D^b_\Lambda(X)^{t=0} \\ \text{} & & \end{tikzcd}$$

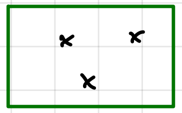
gluing

perverse t-structure

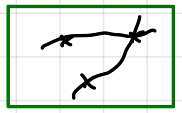
$i_\lambda^! \mathcal{F} \in D_{\mathrm{const}}(X_\lambda)^{t\geq 0}$

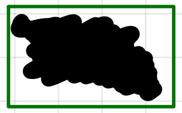
$i_\lambda^* \mathcal{F} \in D_{\mathrm{const}}(X_\lambda)^{t\leq 0}$

$\mathcal{H}^{>0}(\mathcal{F}) = 0$

$\mathcal{H}^{0}(\mathcal{F})$ 

$\vdots$

$\mathcal{H}^{-d_\lambda+1}(\mathcal{F})$ 

$\mathcal{H}^{-d_\lambda}(\mathcal{F})$ 

$\mathcal{H}^{<-d_\lambda}(\mathcal{F}) = 0$

(*) $\mathcal{L} \in \mathrm{Loc}(X_\lambda)$

$\mathbb{D}(\mathcal{L}[d_\lambda]) \overset{\uparrow}{=} \mathcal{L}^\vee[2d_\lambda - d_\lambda] = \mathcal{L}^\vee[d_\lambda]$

X_λ smooth

4. Perverse sheaves

Verdier duality

std. t-structure shifted by d_λ

$$\begin{tikzcd}
\mathbb{D} \overset{(*)}{\circlearrowright} \mathrm{Loc}(X_\lambda)[d_\lambda] \arrow[d, squiggly] \arrow[r, phantom, "="] & D^b_{\mathrm{const}}(X_\lambda)^{t=0} \arrow[d, squiggly] \arrow[dl, phantom, "\text{gluing}"] \\
\mathbb{D} \circlearrowright \mathrm{Perv}_\Lambda(X) \arrow[d, squiggly] \arrow[r, phantom, "="] & D^b_\Lambda(X)^{t=0} \\
\text{ } &
\end{tikzcd}$$

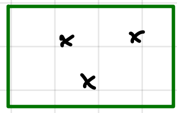
perverse t-structure

$i_\lambda^! \mathcal{F} \in D_{\mathrm{const}}(X_\lambda)^{t\geq 0}$

$i_\lambda^* \mathcal{F} \in D_{\mathrm{const}}(X_\lambda)^{t\leq 0}$

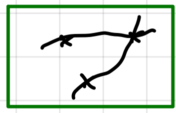
$\mathcal{H}^{>0}(\mathcal{F}) = 0$

$\mathcal{H}^0(\mathcal{F})$

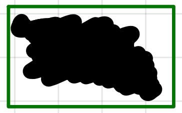


$\vdots$

$\mathcal{H}^{-d_\lambda+1}(\mathcal{F})$



$\mathcal{H}^{-d_\lambda}(\mathcal{F})$



$\mathcal{H}^{<-d_\lambda}(\mathcal{F}) = 0$

dim supp $\mathcal{H}^{-i}(\mathcal{F}) \leq i$

(*) $\mathcal{L} \in \mathrm{Loc}(X_\lambda)$

$\mathbb{D}(\mathcal{L}[d_\lambda]) \underset{\substack{\uparrow \\ X_\lambda \text{ smooth}}}{=} \mathcal{L}^\vee[2d_\lambda - d_\lambda] = \mathcal{L}^\vee[d_\lambda]$

4. Perverse sheaves

Verdier duality

std. t-structure shifted by d_λ

$$\begin{tikzcd} \mathbb{D} \overset{(*)}{\circlearrowright} \mathrm{Loc}(X_\lambda)[d_\lambda] \arrow[r, phantom, "="] \arrow[d, squiggly, hook] \arrow[dr, phantom, "\text{gluing}"] & D^b_{const}(X_\lambda)^{t=0} \arrow[d, squiggly, hook] \\ \mathbb{D} \circlearrowright \mathrm{Perv}_\Lambda(X) \arrow[r, phantom, "="] \arrow[d, squiggly, hook] & D^b_\Lambda(X)^{t=0} \\ \text{ } & \end{tikzcd}$$

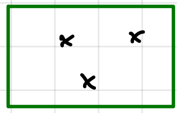
perverse t-structure

$i_\lambda^! \mathcal{F} \in D_{const}(X_\lambda)^{t \geq 0}$

$i_\lambda^* \mathcal{F} \in D_{const}(X_\lambda)^{t \leq 0}$

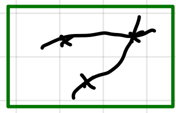
$\mathcal{H}^{>0}(\mathcal{F}) = 0$

$\mathcal{H}^{0}(\mathcal{F})$

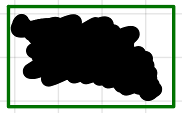


$\vdots$

$\mathcal{H}^{-d_\lambda+1}(\mathcal{F})$



$\mathcal{H}^{-d_\lambda}(\mathcal{F})$



$\mathcal{H}^{<-d_\lambda}(\mathcal{F}) = 0$

dim supp $\mathcal{H}^{-i}(\mathcal{F}) \leq i$

(*) $\mathcal{L} \in \mathrm{Loc}(X_\lambda)$

$\mathbb{D}(\mathcal{L}[d_\lambda]) \underset{\substack{\uparrow \\ X_\lambda \text{ smooth}}}{=} \mathcal{L}^\vee[2d_\lambda - d_\lambda] = \mathcal{L}^\vee[d_\lambda]$

5. Simple perverse sheaves

$\mathcal{L} \in \mathrm{Loc}(X_\lambda)$ $\qquad$ $X_\lambda \xhookrightarrow{j} \overline{X}_\lambda \xhookrightarrow{k} X$

5. Simple perverse sheaves

$$\begin{tikzcd} \mathcal{L} \arrow[d, mapsto] & \in & \mathrm{Loc}(X_\lambda) \arrow[d, "k_! j_{!*}"] \\ IC(\overline{X}_\lambda, \mathcal{L}) & \in & \mathrm{Perv}_\Lambda(X) \end{tikzcd}$$

$$\begin{tikzcd} X_\lambda \arrow[r, hook, "j"] & \overline{X}_\lambda \arrow[r, hook, "k"] & X \end{tikzcd}$$

5. Simple perverse sheaves

$$\begin{tikzcd}
\mathcal{L} \arrow[d, mapsto] & \in & \mathrm{Loc}(X_\lambda) \arrow[d, "k_!\, j_{!*}"] \\
IC(\overline{X}_\lambda, \mathcal{L}) & \in & \mathrm{Perv}_\Lambda(X)
\end{tikzcd}$$

$$X_\lambda \xhookrightarrow{j} \overline{X}_\lambda \xhookrightarrow{k} X$$

minimal extension

$j_{!*} = \mathrm{im}(j_! \to j_*)$

5. Simple perverse sheaves

$$\begin{tikzcd} X_\lambda \arrow[r, hook, "j"] & \overline{X}_\lambda \arrow[r, hook, "k"] & X \end{tikzcd}$$

$$\begin{tikzcd} \mathcal{L} \arrow[d, mapsto] & \in & \mathrm{Loc}(X_\lambda) \arrow[d, "k_! j_{!*}"] \\ IC(\overline{X}_\lambda, \mathcal{L}) & \in & \mathrm{Perv}_\Lambda(X) \end{tikzcd}$$

minimal extension

$j_{!*} = \mathrm{im}(j_! \to j_*)$

Intersection cohomology complex

5. Simple perverse sheaves

$$X_\lambda \xhookrightarrow{j} \overline{X}_\lambda \xhookrightarrow{k} X$$

$$\begin{tikzcd} \mathcal{L} \arrow[d, mapsto] & \in & \mathrm{Loc}(X_\lambda) \arrow[d, "k_! \, j_{!*}"] \\ IC(\overline{X}_\lambda, \mathcal{L}) & \in & \mathrm{Perv}_\Lambda(X) \end{tikzcd}$$

minimal extension

$j_{!*} = \mathrm{im}(j_! \to j_*)$

$\rightsquigarrow \mathcal{H}^i(IC) = IH^i$

Intersection cohomology complex $\rightsquigarrow$ Intersection cohomology

5. Simple perverse sheaves

$$X_\lambda \xhookrightarrow{j} \overline{X}_\lambda \xhookrightarrow{k} X$$

$$\begin{tikzcd}
\mathcal{L} \arrow[d, mapsto] & \in \mathrm{Loc}(X_\lambda) \arrow[d, "k_! \, j_{!*}"] \\
IC(\overline{X}_\lambda, \mathcal{L}) & \in \mathrm{Perv}_\Lambda(X)
\end{tikzcd}$$

minimal extension

$j_{!*} = \mathrm{im}(j_! \to j_*)$

$IC(\overline{X}_\lambda, \mathcal{L}) \in \mathrm{Perv}_\Lambda(X) \rightsquigarrow \mathcal{H}^i(IC) = I\mathcal{H}^i$

Intersection cohomology complex $\rightsquigarrow$ Intersection cohomology

$\mathrm{Irr}(\mathrm{Perv}_\Lambda(X))$

5. Simple perverse sheaves

$$\begin{tikzcd} \mathcal{L} \arrow[d, mapsto] & \in & \mathrm{Loc}(X_\lambda) \arrow[d, "k_! j_{!*}"] \\ IC(\overline{X}_\lambda, \mathcal{L}) & \in & \mathrm{Perv}_\Lambda(X) \end{tikzcd}$$

$$X_\lambda \xhookrightarrow{j} \overline{X}_\lambda \xhookrightarrow{k} X$$

minimal extension $j_{!*} = \mathrm{im}(j_! \to j_*)$

$IC(\overline{X}_\lambda, \mathcal{L}) \in \mathrm{Perv}_\Lambda(X) \rightsquigarrow \mathcal{H}^i(IC) = I\mathcal{H}^i$

Intersection cohomology complex $\rightsquigarrow$ Intersection cohomology

$$\mathrm{Irr}(\mathrm{Perv}_\Lambda(X)) \xleftrightarrow{1:1} \{(\lambda, \mathcal{L}) \mid \mathcal{L} \in \mathrm{Irr}(\mathrm{Loc}(X_\lambda))\}$$

6. Highest weight category

Assume X_λ simply-connected $\Rightarrow \mathrm{Loc}(X_\lambda) = k\text{-mod}$.

6. Highest weight category

Assume X_λ simply-connected $\Rightarrow \mathrm{Loc}(X_\lambda) = k\text{-mod}$. $\lambda \leq \lambda' \iff X_\lambda \subset \overline{X_{\lambda'}}$

6. Highest weight category

Assume X_λ simply-connected $\Rightarrow \mathrm{Loc}(X_\lambda) = k\text{-mod}$. $\lambda \leq \lambda' \iff X_\lambda \subset \overline{X_{\lambda'}}$

$$\{ \underset{\substack{\uparrow \\ \text{simple} \\ IC(X_\lambda, k)}}{L}_\lambda \}$$

6. Highest weight category

Assume X_λ simply-connected $\Rightarrow \mathrm{Loc}(X_\lambda) = k\text{-mod}$. $\lambda \leq \lambda' \iff X_\lambda \subset \overline{X_{\lambda'}}$

standard

$i_{\lambda!}k[d_\lambda]$

$\downarrow$

$\{\Delta_\lambda\}$

$\mathbb{D} \circlearrowright \{L_\lambda\}$

$\uparrow$

simple

$IC(X_\lambda, k)$

6. Highest weight category

Assume X_λ simply-connected $\Rightarrow \mathrm{Loc}(X_\lambda) = k\text{-mod}$. $\lambda \leq \lambda' \iff X_\lambda \subset \overline{X_{\lambda'}}$

$$\{\overset{\substack{\text{standard}\\ i_{\lambda!}k[d_\lambda]\\ \downarrow}}{\Delta_\lambda}\}$$

$$\mathbb{D} \circlearrowright \{\underset{\substack{\uparrow\\ \text{simple}\\ IC(X_\lambda,k)}}{L_\lambda}\}$$

$$\{\underset{\substack{\uparrow\\ \text{costandard}\\ i_{\lambda*}k[d_\lambda]}}{\nabla_\lambda}\}$$

6. Highest weight category

Assume X_λ simply-connected $\Rightarrow \mathrm{Loc}(X_\lambda) = k\text{-mod}$. $\lambda \leq \lambda' \iff X_\lambda \subset \overline{X_{\lambda'}}$

$$\begin{tikzcd} \mathbb{D} \circlearrowright \{ L_\lambda \} \\ \begin{array}{c}\text{simple} \\ IC(X_\lambda, k)\end{array} \arrow[u] \end{tikzcd}$$

$$\begin{tikzcd} \begin{array}{c}\text{standard} \\ i_{\lambda !}k[d_\lambda]\end{array} \arrow[d] \\ \{ \Delta_\lambda \} \arrow[d, leftrightarrow, "\mathbb{D}"] \\ \{ \nabla_\lambda \} \\ \begin{array}{c}\text{costandard} \\ i_{\lambda *}k[d_\lambda]\end{array} \arrow[u] \end{tikzcd}$$

6. Highest weight category

Assume X_λ simply-connected $\Rightarrow \mathrm{Loc}(X_\lambda) = \Bbbk\text{-mod}$. $\lambda \leq \lambda' \Leftrightarrow X_\lambda \subset \overline{X_{\lambda'}}$

$$\begin{tikzcd}
 & \text{standard} \\
 & i_{\lambda !}\Bbbk[d_\lambda] \arrow[d] \\
 & \{\Delta_\lambda\} \arrow[dl] \arrow[dd, leftrightarrow, "\mathbb{D}"] \\
\mathbb{D} \circlearrowright \{L_\lambda\} & \\
\text{simple} \arrow[u] & \{\nabla_\lambda\} \arrow[ul] \\
IC(X_\lambda, \Bbbk) & \text{costandard} \arrow[u] \\
 & i_{\lambda *}\Bbbk[d_\lambda]
\end{tikzcd}$$

projective cover

$$\begin{tikzcd}
P_\lambda \arrow[r, two heads] \arrow[d] & \Delta_\lambda \arrow[r] & L_\lambda \\
\{P_\lambda\} & &
\end{tikzcd}$$

6. Highest weight category

Assume X_λ simply-connected $\Rightarrow \mathrm{Loc}(X_\lambda) = k\text{-mod}$. $\lambda \leq \lambda' \Leftrightarrow X_\lambda \subset \overline{X_{\lambda'}}$

$\mathbb{D} \circlearrowright \{ L_\lambda \}$

$\uparrow$

simple

$IC(X_\lambda, k)$

standard

$i_{\lambda!}k[d_\lambda]$

$\downarrow$

$$\begin{tikzcd}
\{ \Delta_\lambda \} \arrow[d, leftrightarrow, "\mathbb{D}"] \\
\{ \nabla_\lambda \}
\end{tikzcd}$$

$\uparrow$

costandard

$i_{\lambda*}k[d_\lambda]$

projective cover

$P_\lambda \twoheadrightarrow \Delta_\lambda \to L_\lambda$

$\downarrow$

$\{ P_\lambda \}$

$\{ I_\lambda \}$

$\uparrow$

injective hull

$L_\lambda \subset \nabla_\lambda \subset I_\lambda$

6. Highest weight category

Assume X_λ simply-connected $\Rightarrow \mathrm{Loc}(X_\lambda) = k\text{-mod}$. $\lambda \leq \lambda' \iff X_\lambda \subset \overline{X_{\lambda'}}$

$$\begin{tikzcd}
\mathbb{D} \circlearrowright \{ L_\lambda \} \\
\text{simple} \arrow[u] \\
IC(X_\lambda, k)
\end{tikzcd}$$

$$\begin{tikzcd}
\text{standard} \\
i_{\lambda !} k[d_\lambda] \arrow[d] \\
\{ \Delta_\lambda \} \arrow[d, leftrightarrow, "\mathbb{D}"] \\
\{ \nabla_\lambda \} \\
\text{costandard} \arrow[u] \\
i_{\lambda *} k[d_\lambda]
\end{tikzcd}$$

$$\begin{tikzcd}
\text{projective cover} \\
P_\lambda \twoheadrightarrow \Delta_\lambda \rightarrow L_\lambda \arrow[d] \\
\{ P_\lambda \} \arrow[d, leftrightarrow, bend left, "\mathbb{D}"] \\
\{ I_\lambda \} \\
\text{injective hull} \arrow[u] \\
L_\lambda \subset \nabla_\lambda \subset I_\lambda
\end{tikzcd}$$

6. Highest weight category

Assume X_λ simply-connected $\Rightarrow \mathrm{Loc}(X_\lambda) = k\text{-mod}$. $\lambda \leq \lambda' \iff X_\lambda \subset \overline{X_{\lambda'}}$

$$\begin{tikzcd}
\mathbb{D} \circlearrowright \{ L_\lambda \} \\
\text{simple} \arrow[u] \\
IC(X_\lambda, k)
\end{tikzcd}$$

$$\begin{tikzcd}
\text{standard} \\
i_{\lambda!} k[d_\lambda] \arrow[d] \\
\{ \Delta_\lambda \} \arrow[d, leftrightarrow, "\mathbb{D}"] \\
\{ \nabla_\lambda \} \\
\text{costandard} \arrow[u] \\
i_{\lambda*} k[d_\lambda]
\end{tikzcd}$$

$$\begin{tikzcd}
\text{projective cover} \\
P_\lambda \arrow[r, two heads] \arrow[d] & \Delta_\lambda \arrow[r] & L_\lambda \\
\{ P_\lambda \} \arrow[ddd, bend left=60, leftrightarrow, "\mathbb{D}"] \\
\text{tilting} \arrow[d] \\
\{ T_\lambda \} \\
\{ I_\lambda \} \\
\text{injective hull} \arrow[u] \\
L_\lambda \subset \nabla_\lambda \subset I_\lambda
\end{tikzcd}$$

6. Highest weight category

Assume X_λ simply-connected $\Rightarrow \mathrm{Loc}(X_\lambda) = \Bbbk\text{-mod}$. $\lambda \leq \lambda' \Leftrightarrow X_\lambda \subset \overline{X_{\lambda'}}$

$$\begin{tikzcd}
\mathbb{D} \circlearrowright \{ L_\lambda \} \\
\text{simple} \arrow[u] \\
IC(X_\lambda, \Bbbk)
\end{tikzcd}$$

$$\begin{tikzcd}
\text{standard} \\
i_{\lambda!}\Bbbk[d_\lambda] \arrow[d] \\
\{ \Delta_\lambda \} \arrow[d, leftrightarrow, "\mathbb{D}"] \\
\{ \nabla_\lambda \} \\
\text{costandard} \arrow[u] \\
i_{\lambda *}\Bbbk[d_\lambda]
\end{tikzcd}$$

$$\begin{tikzcd}
\text{projective cover} \\
P_\lambda \arrow[r, two heads] \arrow[d] & \Delta_\lambda \arrow[r] & L_\lambda \\
\{ P_\lambda \} \arrow[ddd, bend left, leftrightarrow, "\mathbb{D}"] \\
\text{tilting} \arrow[d] \\
\{ T_\lambda \} \circlearrowleft \mathbb{D} \\
\{ I_\lambda \} \\
\text{injective hull} \arrow[u] \\
L_\lambda \subset \nabla_\lambda \subset I_\lambda
\end{tikzcd}$$

6. Highest weight category

Assume X_λ simply-connected $\Rightarrow \mathrm{Loc}(X_\lambda) = k\text{-mod}$. $\lambda \leq \lambda' \Leftrightarrow X_\lambda \subset \overline{X_{\lambda'}}$

$$\begin{tikzcd}
 & & \text{..admit filtrations with subquotients in...} \arrow[dd] & \text{projective cover} \\
 & \text{standard } i_{\lambda!}k[d_\lambda] \arrow[d] & & P_\lambda \twoheadrightarrow \Delta_\lambda \to L_\lambda \arrow[d] \\
 & \{\Delta_\lambda\} \arrow[dl] \arrow[dd, leftrightarrow, "\mathbb{D}"] & {} & \{P_\lambda\} \arrow[ll] \arrow[ddd, bend left=60, leftrightarrow, "\mathbb{D}"] \\
\mathbb{D} \circlearrowright \{L_\lambda\} & & & \text{tilting} \arrow[d] \\
 & \{\nabla_\lambda\} \arrow[ul] & & \{T_\lambda\} \circlearrowleft \mathbb{D} \arrow[uull] \arrow[ll] \\
\text{simple} \arrow[uu] & \text{costandard } i_{\lambda*}k[d_\lambda] \arrow[u] & & \{I_\lambda\} \arrow[ull] \\
IC(X_\lambda, k) & & & \text{injective hull} \arrow[u] \\
 & & & L_\lambda \subset \nabla_\lambda \subset I_\lambda
\end{tikzcd}$$

6. Highest weight category

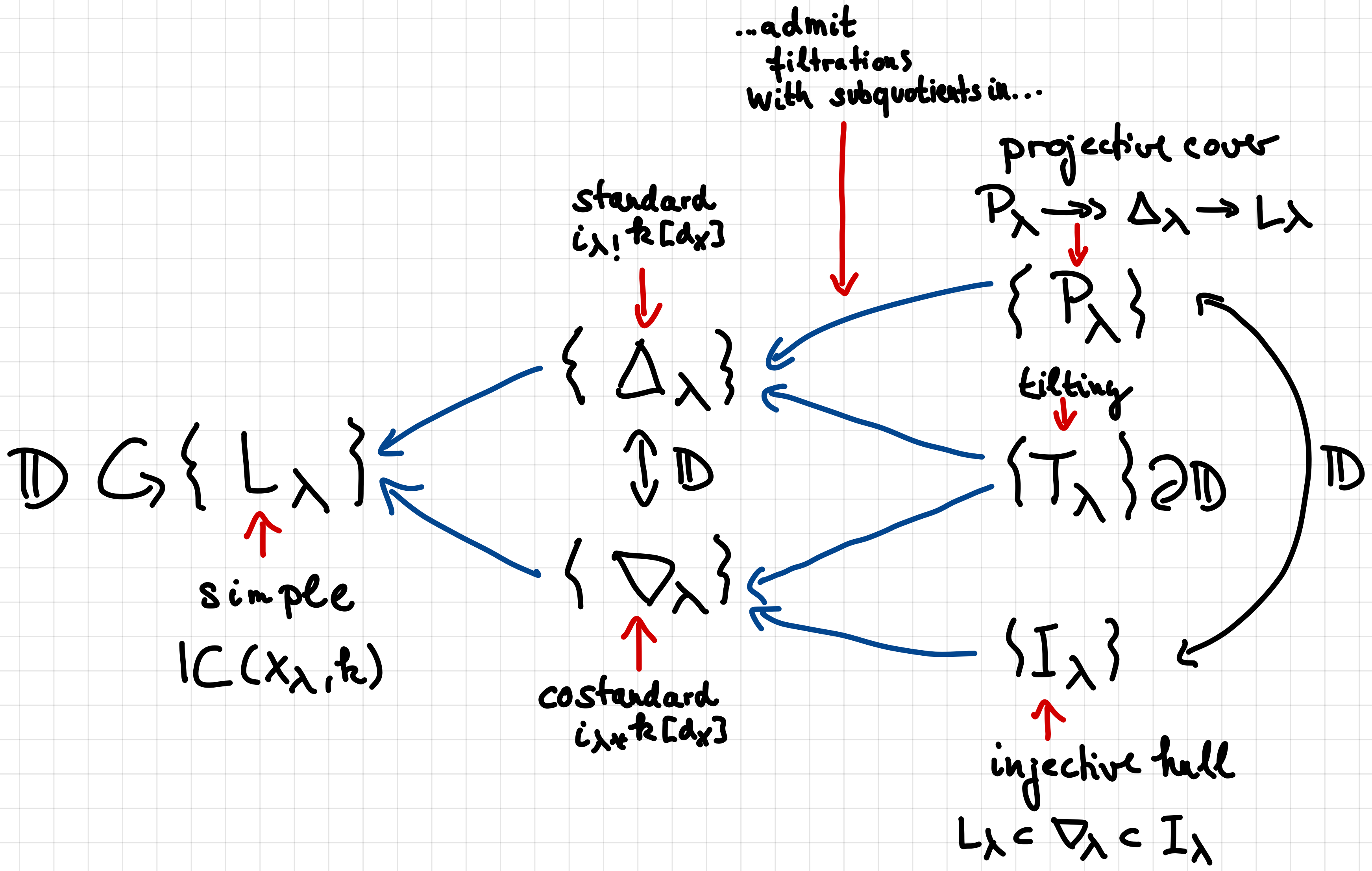
Assume X_λ simply-connected $\Rightarrow \mathrm{Loc}(X_\lambda) = \Bbbk\text{-mod}$. $\lambda \leq \lambda' \Leftrightarrow X_\lambda \subset \overline{X_{\lambda'}}$

$$\begin{tikzcd}
 & \text{standard } i_{\lambda !}\Bbbk[d_\lambda] \arrow[d] & \text{...admit filtrations with subquotients in...} \arrow[dl] & \text{projective cover } P_\lambda \twoheadrightarrow \Delta_\lambda \to L_\lambda \arrow[d] \\
 & \{\Delta_\lambda\} \arrow[ddl] \arrow[ddd, leftrightarrow, "\mathbb{D}"] & & \{P_\lambda\} \arrow[ll] \arrow[ddd, bend left, leftrightarrow, "\mathbb{D}"] \\
 & & & \text{tilting} \arrow[d] \\
\mathbb{D} \circlearrowright \{L_\lambda\} & & & \{T_\lambda\} \circlearrowleft \mathbb{D} \arrow[uull] \arrow[dll] \\
\text{simple } IC(X_\lambda, \Bbbk) \arrow[u] & \{\nabla_\lambda\} \arrow[ul] & & \{I_\lambda\} \arrow[ll] \\
 & \text{costandard } i_{\lambda *}\Bbbk[d_\lambda] \arrow[u] & & \text{injective hull } L_\lambda \subset \nabla_\lambda \subset I_\lambda \arrow[u]
\end{tikzcd}$$

$$[\Delta_\lambda : L_{\lambda'}] = (P_{\lambda'} : \Delta_\lambda)$$

6. Highest weight category

Assume X_λ simply-connected $\Rightarrow \mathrm{Loc}(X_\lambda) = k\text{-mod}$. $\lambda \leq \lambda' \Leftrightarrow X_\lambda \subset \overline{X_{\lambda'}}$

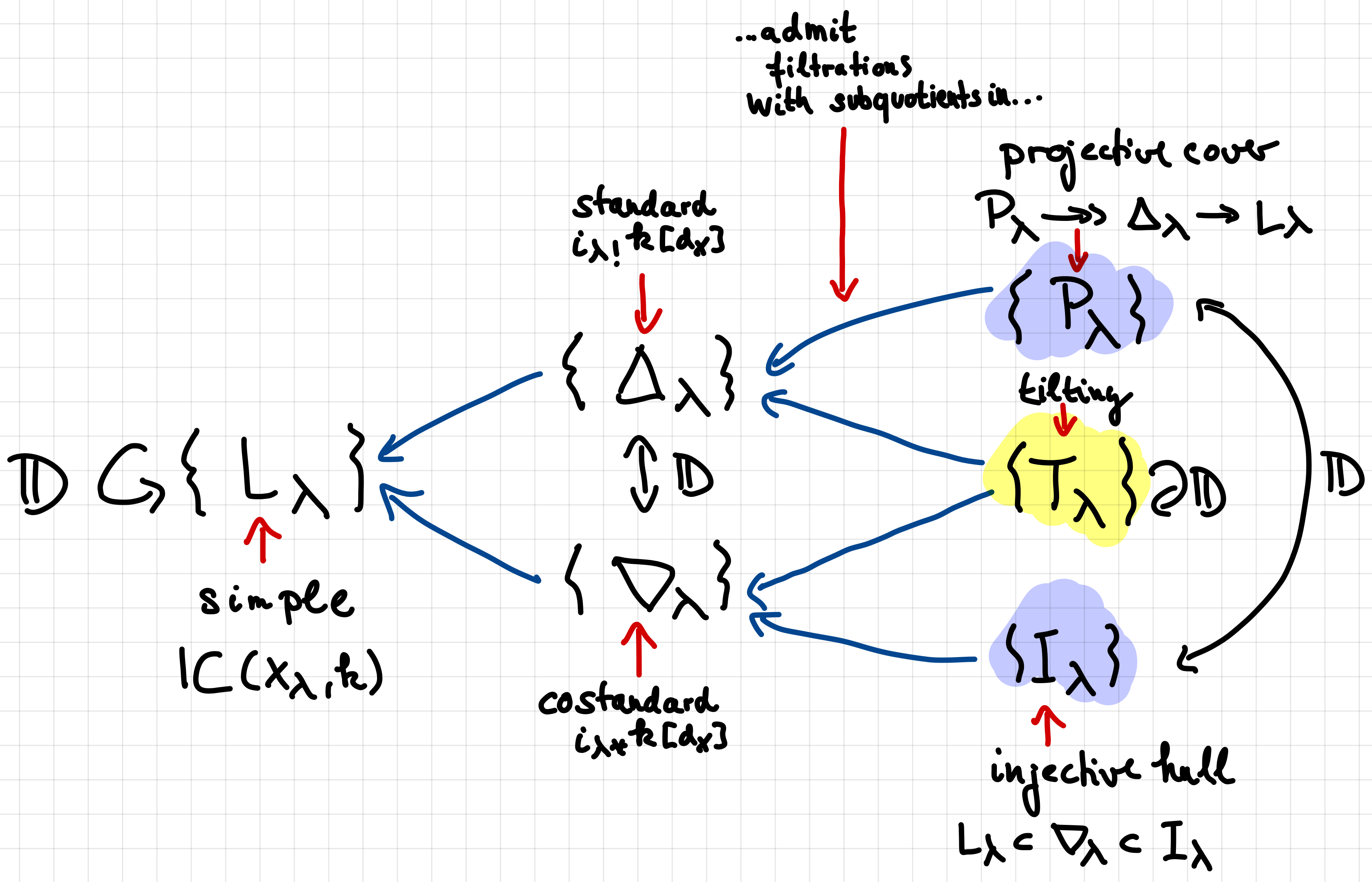


$\rightsquigarrow$ bases for $K_0(\mathrm{Perv}_\Lambda(X))$
$\cong$
$\mathbb{Z}\Lambda$

$$[\Delta_\lambda : L_{\lambda'}] = (P_{\lambda'} : \Delta_\lambda)$$

6. Highest weight category

Assume X_λ simply-connected $\Rightarrow \mathrm{Loc}(X_\lambda) = k\text{-mod}$. $\lambda \leq \lambda' \iff X_\lambda \subset \overline{X_{\lambda'}}$



$\rightsquigarrow$ bases for $K_0(\mathrm{Perv}_\Lambda(X))$

$\cong$

$\mathbb{Z}\Lambda$

$$[\Delta_\lambda : L_{\lambda'}] = (P_{\lambda'} : \Delta_\lambda)$$

🟦 Λ finite

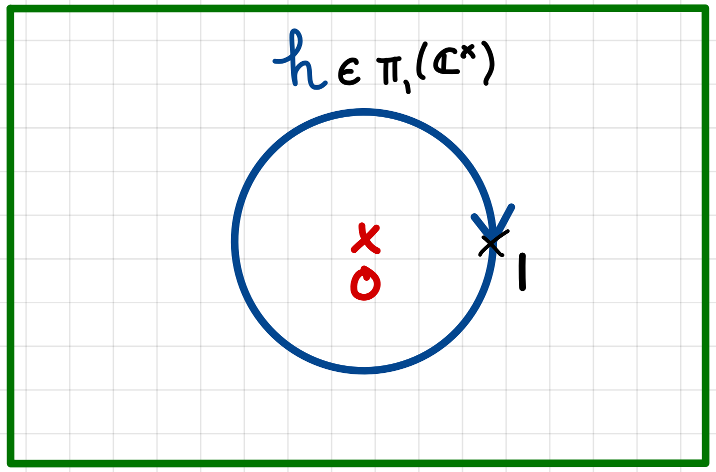
🟨 $\Lambda = \bigcup_{i=0}^{\infty}$ finite

7. Example!

$$X = \mathbb{C} = \{0\} \uplus \mathbb{C}^\times$$

7. Example!

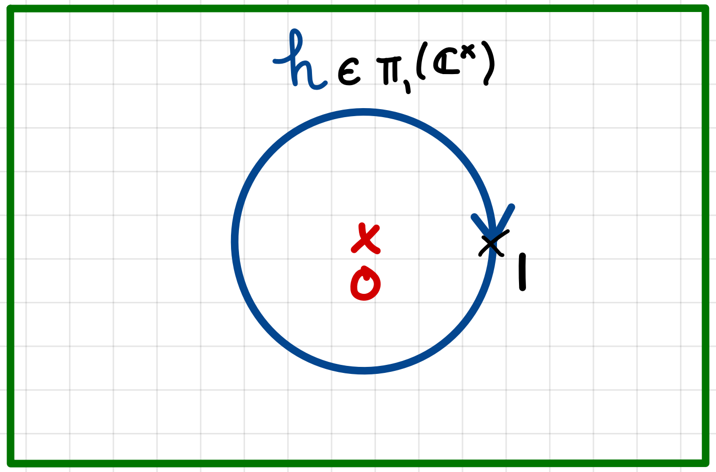
$$X = \mathbb{C} = \{0\} \uplus \mathbb{C}^\times$$



7. Example!

$$X = \mathbb{C} = \{0\} \uplus \mathbb{C}^{\times}$$

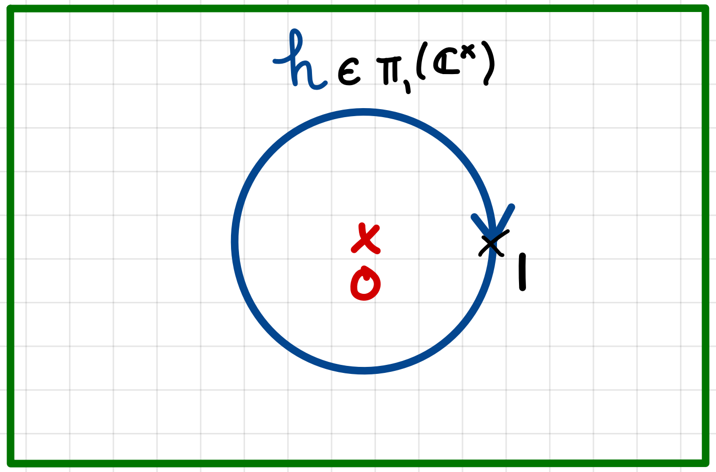
$$\mathcal{F} \in \mathrm{Perv}_{\Lambda}(X)$$



7. Example!

$$X = \mathbb{C} = \{0\} \uplus \mathbb{C}^\times$$

$$\mathcal{F} \in \mathrm{Perv}_\Lambda(X)$$

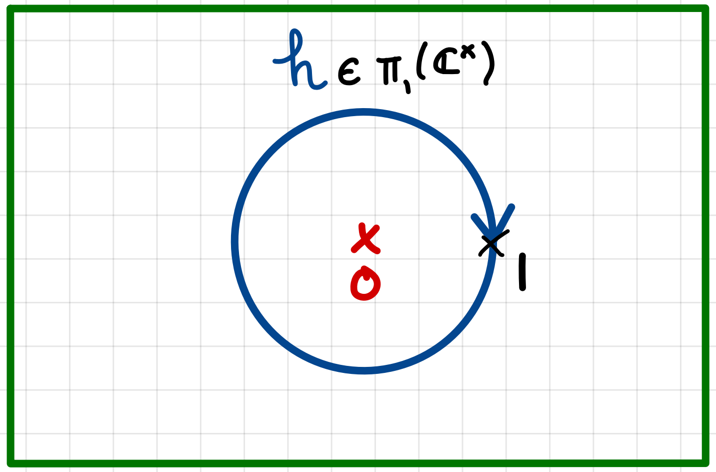


$$0 \to H^{-1}(\mathbb{C};\mathcal{F}) \to H^{-1}(\{1\};\mathcal{F}) \xrightarrow{\text{can}} H^0(\mathbb{C},\{1\};\mathcal{F}) \to H^0(\mathbb{C};\mathcal{F}) \to 0$$

7. Example!

$$X = \mathbb{C} = \{0\} \uplus \mathbb{C}^{\times}$$

$$\mathcal{F} \in \mathrm{Perv}_{\Lambda}(X)$$

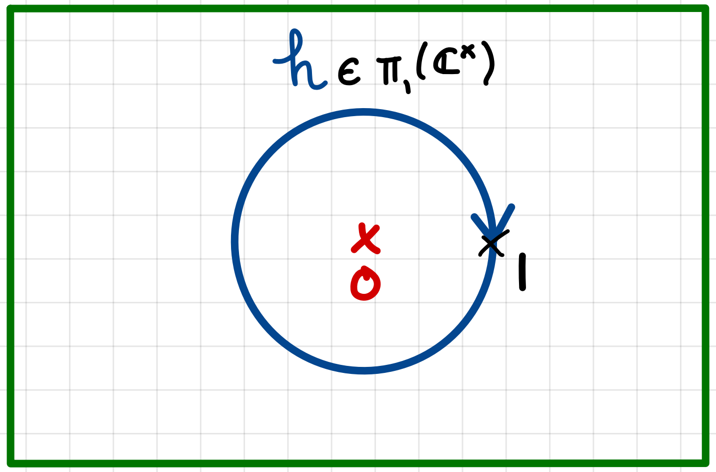


$$\begin{tikzcd}
 & H^{-1}(\{0\};\mathcal{F}) \arrow[d, equal] \arrow[dr, "sp^*"] & & & & \\
0 \arrow[r] & H^{-1}(\mathbb{C};\mathcal{F}) \arrow[r] & H^{-1}(\{1\};\mathcal{F}) \arrow[r, "can"] & H^{0}(\mathbb{C}, \{1\};\mathcal{F}) \arrow[r] & H^{0}(\mathbb{C};\mathcal{F}) \arrow[r] & 0
\end{tikzcd}$$

7. Example!

$$X = \mathbb{C} = \{0\} \uplus \mathbb{C}^{\times}$$

$$\mathcal{F} \in \mathrm{Perv}_{\Lambda}(X)$$

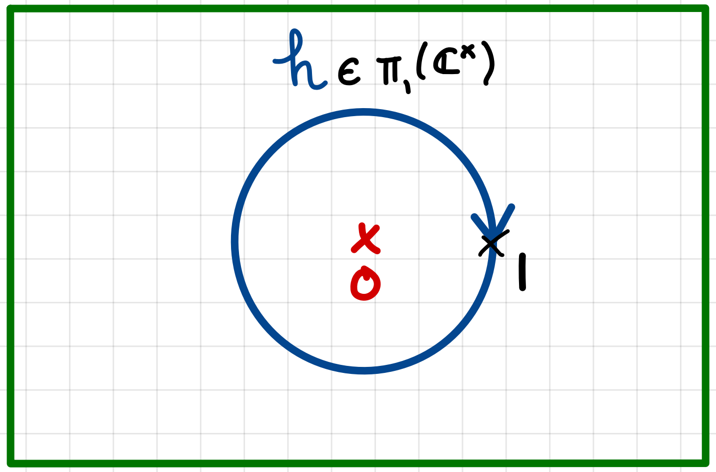


$$\begin{tikzcd}
 & H^{-1}(\{0\};\mathcal{F}) \arrow[d, equal] \arrow[dr, "sp^*"] & & & & \\
0 \arrow[r] & H^{-1}(\mathbb{C};\mathcal{F}) \arrow[r] \arrow[d] & H^{-1}(\{1\};\mathcal{F}) \arrow[r, "can"] \arrow[d, "h-1"] & H^{0}(\mathbb{C},\{1\};\mathcal{F}) \arrow[r] & H^{0}(\mathbb{C};\mathcal{F}) \arrow[r] \arrow[d] & 0 \\
 & 0 \arrow[r] & H^{-1}(\{1\};\mathcal{F}) \arrow[r, "id"] & H^{-1}(\{1\};\mathcal{F}) \arrow[r] & 0 &
\end{tikzcd}$$

7. Example!

$$X = \mathbb{C} = \{0\} \uplus \mathbb{C}^\times$$

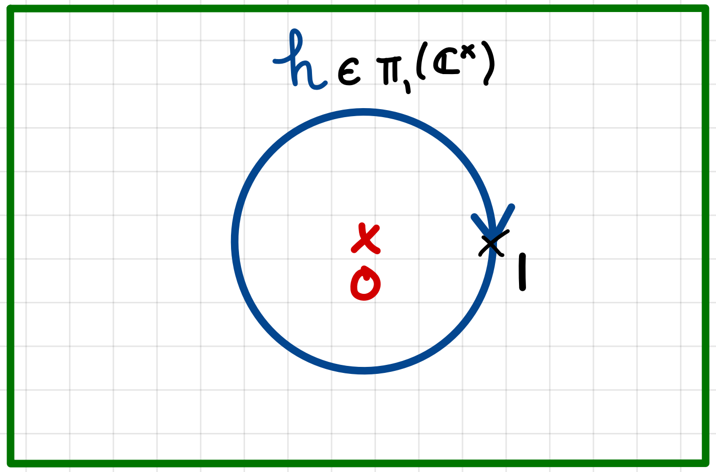
$$\mathcal{F} \in \mathrm{Perv}_\Lambda(X)$$



$$\begin{tikzcd}
 & H^{-1}(\{0\};\mathcal{F}) \arrow[d, equal] \arrow[dr, "sp^*"] & & & & \\
0 \arrow[r] & H^{-1}(\mathbb{C};\mathcal{F}) \arrow[r] \arrow[d] & H^{-1}(\{1\};\mathcal{F}) \arrow[r, "can"] \arrow[d, "h-1"] & H^0(\mathbb{C},\{1\};\mathcal{F}) \arrow[r] \arrow[d, dashed, "var"] & H^0(\mathbb{C};\mathcal{F}) \arrow[r] \arrow[d] & 0 \\
 & 0 \arrow[r] & H^{-1}(\{1\};\mathcal{F}) \arrow[r, "id"] & H^{-1}(\{1\};\mathcal{F}) \arrow[r] & 0 &
\end{tikzcd}$$

7. Example!

$$X = \mathbb{C} = \{0\} \uplus \mathbb{C}^{\times}$$



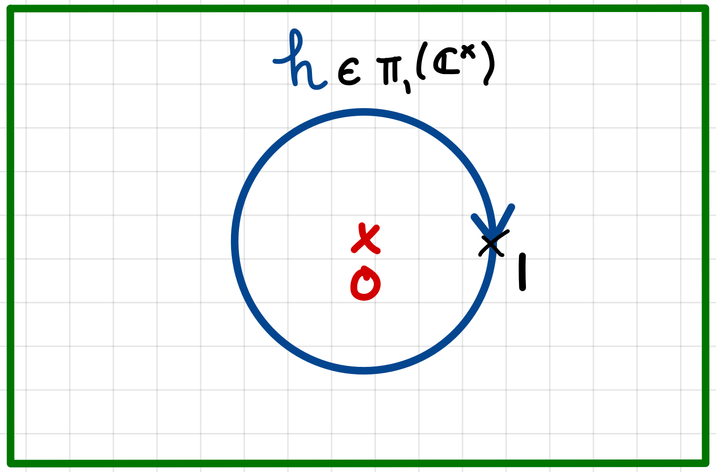
$$\mathcal{F} \in \mathrm{Perv}_{\Lambda}(X)$$

$$\begin{tikzcd}
 & H^{-1}(\{0\};\mathcal{F}) \arrow[d, equal] \arrow[dr, "sp^*"] & \psi \arrow[d, equal] \arrow[r, shift left, "can"] & \varphi \arrow[d, equal] \arrow[l, shift left, "var"] & & \\
0 \arrow[r] & H^{-1}(\mathbb{C};\mathcal{F}) \arrow[r] \arrow[d] & H^{-1}(\{1\};\mathcal{F}) \arrow[r, "can"] \arrow[d, "h-1"] & H^{0}(\mathbb{C},\{1\};\mathcal{F}) \arrow[r] \arrow[d, dashed, "var"] & H^{0}(\mathbb{C};\mathcal{F}) \arrow[r] \arrow[d] & 0 \\
 & 0 \arrow[r] & H^{-1}(\{1\};\mathcal{F}) \arrow[r, "id"] & H^{-1}(\{1\};\mathcal{F}) \arrow[r] & 0 &
\end{tikzcd}$$

7. Example!

$$X = \mathbb{C} = \{0\} \uplus \mathbb{C}^{\times}$$

$$\mathcal{F} \in \mathrm{Perv}_{\Lambda}(X)$$

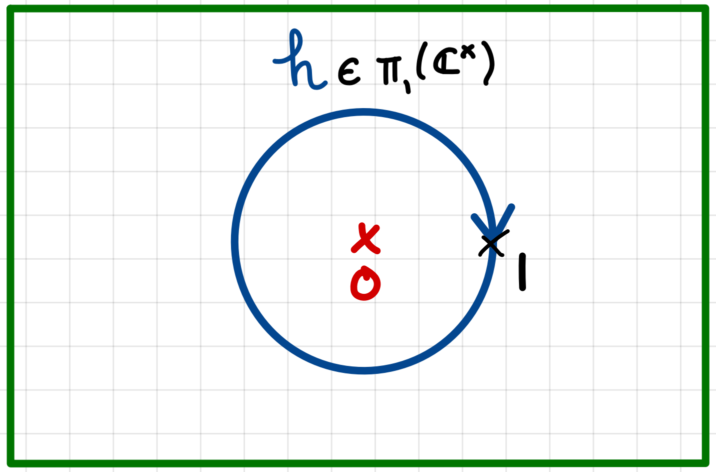


$$\begin{tikzcd}
& H^{-1}(\{0\};\mathcal{F}) \arrow[d, equal] \arrow[dr, "sp^*"] & \psi \arrow[d, equal] \arrow[r, shift left, "\mathrm{can}"] & \varphi \arrow[d, equal] \arrow[l, shift left, "\mathrm{var}"] & & \\
0 \arrow[r] & H^{-1}(\mathbb{C};\mathcal{F}) \arrow[r] \arrow[d] & H^{-1}(\{1\};\mathcal{F}) \arrow[r, "\mathrm{can}"] \arrow[d, "h-1"] & H^{0}(\mathbb{C},\{1\};\mathcal{F}) \arrow[r] \arrow[d, dashed, "\mathrm{var}"] & H^{0}(\mathbb{C};\mathcal{F}) \arrow[r] \arrow[d] & 0 \\
& 0 \arrow[r] & H^{-1}(\{1\};\mathcal{F}) \arrow[r, "\mathrm{id}"] & H^{-1}(\{1\};\mathcal{F}) \arrow[r] & 0 &
\end{tikzcd}$$

$\mathrm{var}\,\mathrm{can} = h - 1$

7. Example!

$$X = \mathbb{C} = \{0\} \uplus \mathbb{C}^{\times}$$



$$\mathcal{F} \in \mathrm{Perv}_{\Lambda}(X)$$

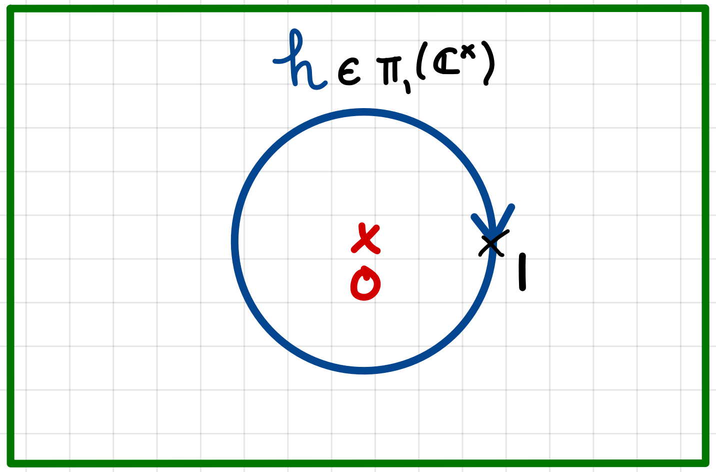
$$\begin{tikzcd}
 & H^{-1}(\{0\};\mathcal{F}) \arrow[d, equal] \arrow[rd, "sp^*"] & \psi \arrow[r, shift left, "\mathrm{can}"] \arrow[d, equal] & \varphi \arrow[l, shift left, "\mathrm{var}"] \arrow[d, equal] & & \\
0 \arrow[r] & H^{-1}(\mathbb{C};\mathcal{F}) \arrow[r] \arrow[d] & H^{-1}(\{1\};\mathcal{F}) \arrow[r, "\mathrm{can}"] \arrow[d, "h-1"] & H^{0}(\mathbb{C},\{1\};\mathcal{F}) \arrow[r] \arrow[d, dashed, "\mathrm{var}"] & H^{0}(\mathbb{C};\mathcal{F}) \arrow[r] \arrow[d] & 0 \\
 & 0 \arrow[r] & H^{-1}(\{1\};\mathcal{F}) \arrow[r, "\mathrm{id}"] & H^{-1}(\{1\};\mathcal{F}) \arrow[r] & 0 &
\end{tikzcd}$$

$\mathrm{var}\,\mathrm{can} = h - 1$

$$\mathrm{Perv}_{\Lambda}(X)$$

7. Example!

$$X = \mathbb{C} = \{0\} \uplus \mathbb{C}^\times$$

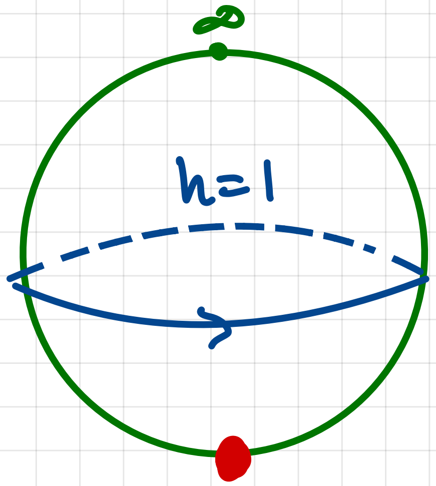


$$\mathcal{F} \in \mathrm{Perv}_\Lambda(X)$$

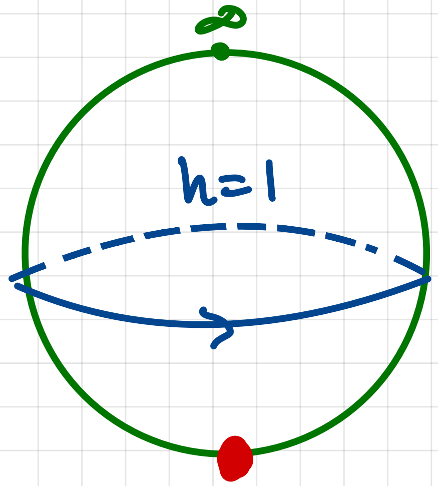
$$\begin{tikzcd}
 & H^{-1}(\{0\};\mathcal{F}) \arrow[d, equal] \arrow[dr, "sp^*"] & \psi \arrow[r, shift left, "can"] \arrow[d, equal] & \varphi \arrow[l, shift left, "var"] \arrow[d, equal] & \mathrm{var}\,\mathrm{can} = h - 1 & \\
0 \arrow[r] & H^{-1}(\mathbb{C};\mathcal{F}) \arrow[r] \arrow[d] & H^{-1}(\{1\};\mathcal{F}) \arrow[r, "can"] \arrow[d, "h-1"] & H^0(\mathbb{C},\{1\};\mathcal{F}) \arrow[r] \arrow[d, dashed, "var"] & H^0(\mathbb{C};\mathcal{F}) \arrow[r] \arrow[d] & 0 \\
 & 0 \arrow[r] & H^{-1}(\{1\};\mathcal{F}) \arrow[r, "id"] & H^{-1}(\{1\};\mathcal{F}) \arrow[r] & 0 &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Perv}_\Lambda(X) \arrow[r, "\sim"] & \{\psi \arrow[rr, shift left, "can"] & & \varphi \arrow[ll, shift left, "var"] \mid h = \mathrm{var}\,\mathrm{can} + 1 \in GL(\psi)\} \\
 & & k\text{-mod} \arrow[ul] \arrow[u] \arrow[ur] &
\end{tikzcd}$$

$$X = \mathbb{P}^1_{\mathbb{C}} = \overset{X_e}{\{0\}} \uplus \overset{X_s}{\mathbb{C}} = \mathbb{C}^{\times} \cup \{\infty\}$$

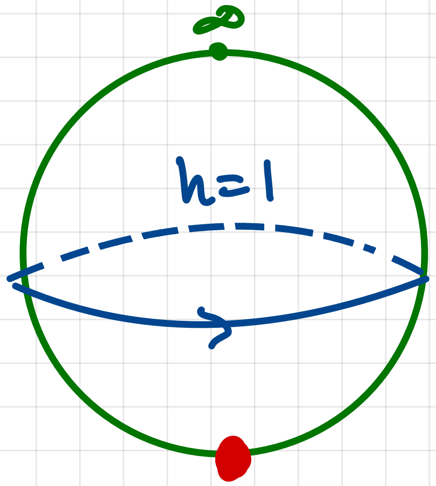


$$X = \mathbb{P}^1_{\mathbb{C}} = \overset{X_e}{\{0\}} \uplus \overset{X_s}{\mathbb{C}} = \mathbb{C}^\times \cup \{\infty\}$$



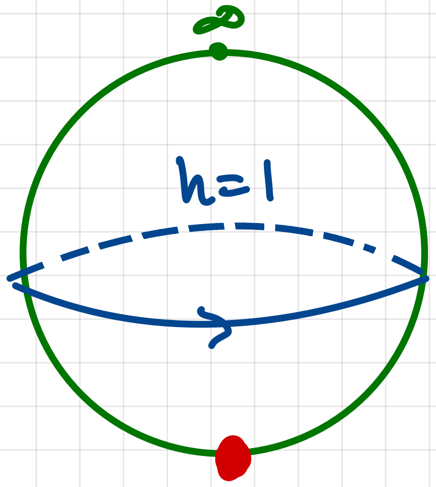
$$\mathrm{Perv}_\Lambda(\mathbb{P}^1_{\mathbb{C}}) \subset \mathrm{Perv}_\Lambda(\mathbb{C})$$

$$X = \mathbb{P}^1_{\mathbb{C}} = \overset{X_e}{\{0\}} \uplus \overset{X_s}{\mathbb{C}} = \mathbb{C}^\times \cup \{\infty\}$$



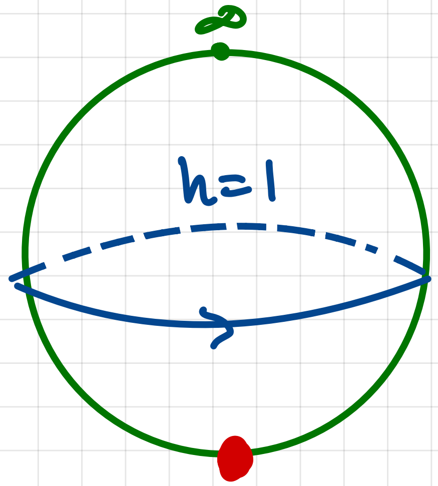
$$\begin{tikzcd}
\mathrm{Perv}_\Lambda(\mathbb{P}^1_{\mathbb{C}}) \subset \mathrm{Perv}_\Lambda(\mathbb{C}) \arrow[d, "\wr"] \\
\left\{ \psi \arrow[r, shift left, "can"] & \varphi \arrow[l, shift left, "var"] \;\middle|\; var\,can = 0 \right\}
\end{tikzcd}$$

$$X = \mathbb{P}^1_{\mathbb{C}} = \overset{X_e}{\{0\}} \uplus \overset{X_s}{\mathbb{C}} = \mathbb{C}^\times \cup \{\infty\}$$



$$\begin{tikzcd} \mathrm{Perv}_\Lambda(\mathbb{P}^1_{\mathbb{C}}) \arrow[d, "\wr"] \arrow[r, phantom, "\subset"] & \mathrm{Perv}_\Lambda(\mathbb{C}) \\ \{\psi \arrow[r, shift left, "can"] & \varphi \arrow[l, shift left, "var"] \mid var\,can = 0\} \; \Big(\xleftarrow{\sim} \; \mathcal{O}_0(\mathfrak{sl}_2(\mathbb{C}))\Big) \end{tikzcd}$$

$$X = \mathbb{P}^1_{\mathbb{C}} = \overset{X_e}{\{0\}} \uplus \overset{X_s}{\mathbb{C}} = \mathbb{C}^\times \cup \{\infty\}$$

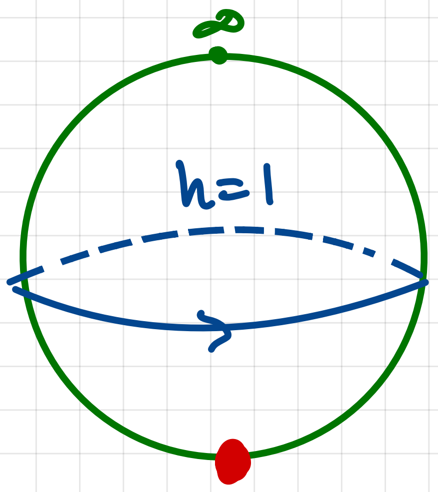


$$\mathrm{Perv}_\Lambda(\mathbb{P}^1_{\mathbb{C}}) \subset \mathrm{Perv}_\Lambda(\mathbb{C})$$

$$\downarrow\wr$$

$$\left\{ \psi \underset{var}{\overset{can}{\rightleftarrows}} \varphi \;\middle|\; var\,can = 0 \right\} \left(\xleftarrow{\sim} \mathcal{O}_0(sl_2(\mathbb{C})) \rightsquigarrow M_0 \underset{E=\begin{pmatrix}0&1\\0&0\end{pmatrix}}{\overset{F=\begin{pmatrix}0&0\\1&0\end{pmatrix}}{\rightleftarrows}} M_{-2} \right)$$

$$X = \mathbb{P}^1_{\mathbb{C}} = \overset{X_e}{\{0\}} \uplus \overset{X_s}{\mathbb{C} = \mathbb{C}^\times \cup \{\infty\}}$$

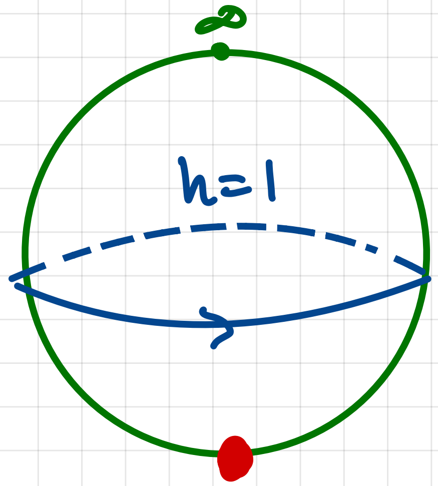


$$\begin{tikzcd} \mathrm{Perv}_\Lambda(\mathbb{P}^1_{\mathbb{C}}) \arrow[d, "\wr"] \arrow[r, phantom, "\subset"] & \mathrm{Perv}_\Lambda(\mathbb{C}) \\ \{\psi \arrow[r, shift left, "can"] & \varphi \arrow[l, shift left, "var"] \mid \mathrm{var}\,\mathrm{can} = 0\} \end{tikzcd}$$

$$\left(\xleftarrow{\simeq} \mathcal{O}_0(\mathfrak{sl}_2(\mathbb{C})) \rightsquigarrow M_0 \underset{E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}{\overset{F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}{\rightleftarrows}} M_{-2} \right)$$

$0 \rightleftarrows \mathbb{C}$	$k_{\{0\}}$	$L_e = \Delta_e = \nabla_e = T_e$

$$X = \mathbb{P}^1_{\mathbb{C}} = \overset{X_e}{\{0\}} \uplus \overset{X_s}{\mathbb{C} = \mathbb{C}^\times \cup \{\infty\}}$$

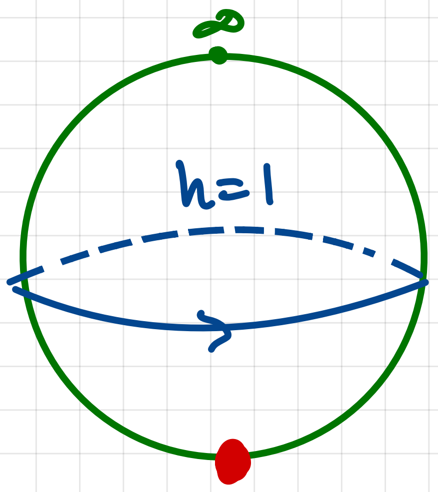


$$\begin{tikzcd} \mathrm{Perv}_\Lambda(\mathbb{P}^1_{\mathbb{C}}) \arrow[d, "\wr"] \subset \mathrm{Perv}_\Lambda(\mathbb{C}) \\ \left\{ \psi \arrow[r, shift left, "can"] & \varphi \arrow[l, shift left, "var"] \mid var\,can = 0 \right\} \end{tikzcd}$$

$$\left(\xleftarrow{\sim} \mathcal{O}_0(\mathfrak{sl}_2(\mathbb{C})) \ni M_0 \underset{E=\begin{pmatrix}0&1\\0&0\end{pmatrix}}{\overset{F=\begin{pmatrix}0&0\\1&0\end{pmatrix}}{\rightleftarrows}} M_{-2} \right)$$

$0 \rightleftarrows \mathbb{C}$	$k_{\{0\}}$	$L_e = \Delta_e = \nabla_e = T_e$
$\mathbb{C} \rightleftarrows 0$	$k_X[1]$	L_s

$$X = \mathbb{P}^1_{\mathbb{C}} = \overset{X_e}{\{0\}} \uplus \overset{X_s}{\mathbb{C}} = \mathbb{C}^\times \cup \{\infty\}$$



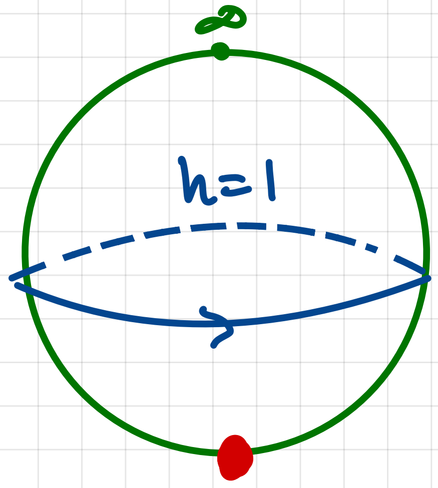
$$\mathrm{Perv}_\Lambda(\mathbb{P}^1_{\mathbb{C}}) \subset \mathrm{Perv}_\Lambda(\mathbb{C})$$

$$\Big\downarrow \wr$$

$$\left\{ \psi \underset{\mathrm{var}}{\overset{\mathrm{can}}{\rightleftarrows}} \varphi \;\middle|\; \mathrm{var}\,\mathrm{can} = 0 \right\} \left(\overset{\sim}{\leftarrow} \mathcal{O}_0(\mathfrak{sl}_2(\mathbb{C})) \rightsquigarrow M_0 \underset{E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}{\overset{F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}{\rightleftarrows}} M_{-2} \right)$$

$0 \rightleftarrows \mathbb{C}$	$k_{\{0\}}$	$L_e = \Delta_e = \nabla_e = T_e$
$\mathbb{C} \rightleftarrows 0$	$k_X[1]$	L_s
$\mathbb{C} \underset{1}{\overset{0}{\rightleftarrows}} \mathbb{C}$	$i_{s!} k_{X_s}[1]$	$\Delta_s = P_s$

$$X = \mathbb{P}^1_\mathbb{C} = \overset{X_e}{\{0\}} \uplus \overset{X_s}{\mathbb{C}} = \mathbb{C}^\times \cup \{\infty\}$$

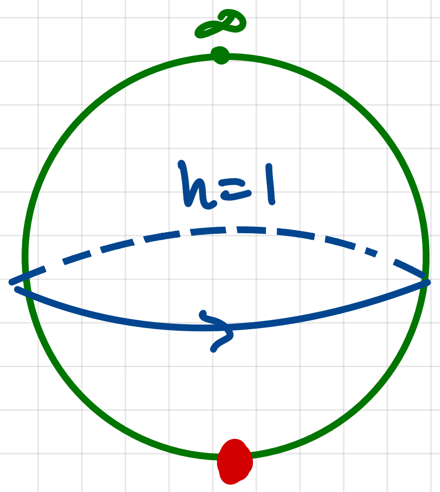


$$\begin{tikzcd}
\mathrm{Perv}_\Lambda(\mathbb{P}^1_\mathbb{C}) \arrow[d, "\wr"] \arrow[r, phantom, "\subset"] & \mathrm{Perv}_\Lambda(\mathbb{C}) \\
\left\{\psi \arrow[r, shift left, "\mathrm{can}"] & \varphi \arrow[l, shift left, "\mathrm{var}"] \mid \mathrm{var}\,\mathrm{can} = 0\right\}
\end{tikzcd}$$

$$\left(\xleftarrow{\simeq} \mathcal{O}_0(\mathfrak{sl}_2(\mathbb{C})) \rightsquigarrow M_0 \underset{E=\begin{pmatrix}0&1\\0&0\end{pmatrix}}{\overset{F=\begin{pmatrix}0&0\\1&0\end{pmatrix}}{\rightleftarrows}} M_{-2} \right)$$

$0 \rightleftarrows \mathbb{C}$	$k_{\{0\}}$	$L_e = \Delta_e = \nabla_e = T_e$
$\mathbb{C} \rightleftarrows 0$	$k_X[1]$	L_s
$\mathbb{C} \underset{1}{\overset{0}{\rightleftarrows}} \mathbb{C}$	$i_{s!}k_{X_s}[1]$	$\Delta_s = P_s$
$\mathbb{C} \underset{0}{\overset{1}{\rightleftarrows}} \mathbb{C}$	$i_{s*}k_{X_s}[1]$	$\nabla_s = I_s$

$$X = \mathbb{P}^1_{\mathbb{C}} = \overset{X_e}{\{0\}} \uplus \overset{X_s}{\mathbb{C}} = \mathbb{C}^\times \cup \{\infty\}$$



$$\mathrm{Perv}_\Lambda(\mathbb{P}^1_{\mathbb{C}}) \subset \mathrm{Perv}_\Lambda(\mathbb{C})$$

$$\big\downarrow \wr$$

$$\left\{ \psi \underset{\mathrm{var}}{\overset{\mathrm{can}}{\rightleftarrows}} \varphi \;\middle|\; \mathrm{var}\,\mathrm{can} = 0 \right\} \left(\xleftarrow{\sim} \mathcal{O}_0(\mathfrak{sl}_2(\mathbb{C})) \rightsquigarrow M_0 \underset{E = \begin{pmatrix}0&1\\0&0\end{pmatrix}}{\overset{F = \begin{pmatrix}0&0\\1&0\end{pmatrix}}{\rightleftarrows}} M_{-2} \right)$$

$0 \rightleftarrows \mathbb{C}$	$k_{\{0\}}$	$L_e = \Delta_e = \nabla_e = T_e$
$\mathbb{C} \rightleftarrows 0$	$k_X[1]$	L_s
$\mathbb{C} \underset{1}{\overset{0}{\rightleftarrows}} \mathbb{C}$	$i_{s!}k_{X_s}[1]$	$\Delta_s = P_s$
$\mathbb{C} \underset{0}{\overset{1}{\rightleftarrows}} \mathbb{C}$	$i_{s*}k_{X_s}[1]$	$\nabla_s = I_s$
$\mathbb{C} \underset{-}{\overset{\Delta}{\rightleftarrows}} \mathbb{C}^2$	$\Xi(k_{X_s}[1])$	$P_e = I_e = T_s$

8 Advanced topics

⚠ $\mathrm{char}\, k = 0$

---

8 Advanced topics

⚠ char $k = 0$

$$\mathrm{Pure}(X) = \langle \, \mathrm{IC} \, \rangle_{\oplus, [1]} \subset D^b(X)$$

8 Advanced topics

⚠ Char $k = 0$

$$\mathrm{Pure}(X) = \langle \mathrm{IC} \rangle_{\oplus, [1]} \subset D^b(X) \quad \left(\mathrm{Pure} \approx D^b_{mix}(X)^{w=0} \approx \mathrm{Chow}(X) \right)$$

8 Advanced topics

⚠ char $k = 0$

$$\mathrm{Pure}(X) = \langle \, \mathrm{IC} \, \rangle_{\oplus, [1]} \subset D^b(X) \quad \left(\mathrm{Pure} \approx D^b_{mix}(X)^{w=0} \approx \mathrm{Chow}(X) \right)$$

Decomposition thm.:

$\pi : X \to Y$ proper

8 Advanced topics

⚠ char $k = 0$

$$\mathrm{Pure}(X) = \langle \mathrm{IC} \rangle_{\oplus, [1]} \subset D^b(X) \quad \left(\mathrm{Pure} \approx D^b_{\mathrm{mix}}(X)^{w=0} \approx \mathrm{Chow}(X) \right)$$

Decomposition thm.:

$\pi : X \to Y$ proper $\Rightarrow$ $\pi_! : \mathrm{Pure}(X) \to \mathrm{Pure}(Y)$

8 Advanced topics

⚠ Char $k = 0$

$$\mathrm{Pure}(X) = \langle \mathrm{IC} \rangle_{\oplus, [1]} \subset D^b(X) \quad \left(\mathrm{Pure} \approx D^b_{mix}(X)^{w=0} \approx \mathrm{Chow}(X) \right)$$

Decomposition thm.:

$\pi : X \to Y$ proper $\Longrightarrow$ $\pi_! : \mathrm{Pure}(X) \to \mathrm{Pure}(Y)$

$$\begin{tikzcd} X \arrow[d, "f"] \\ \mathbb{C} \end{tikzcd}$$

8 Advanced topics

⚠ Char $k = 0$

$$\mathrm{Pure}(X) = \langle \mathrm{IC} \rangle_{\oplus, [1]} \subset D^b(X) \quad \left(\mathrm{Pure} \approx D^b_{mix}(X)^{w=0} \approx \mathrm{Chow}(X) \right)$$

Decomposition thm.:

$\pi : X \to Y$ proper $\implies$ $\pi_! : \mathrm{Pure}(X) \to \mathrm{Pure}(Y)$

$$\begin{tikzcd}
Z \arrow[r] \arrow[d] & X \arrow[d, "f"] & U \arrow[l, "j^*"'] \arrow[d] \\
\{0\} \arrow[r] & \mathbb{C} & \mathbb{C}^\times \arrow[l]
\end{tikzcd}$$

8 Advanced topics

⚠ Char $k = 0$

$$\mathrm{Pure}(X) = \langle \mathrm{IC} \rangle_{\oplus, [1]} \subset D^b(X) \quad \left(\mathrm{Pure} \approx D^b_{mix}(X)^{w=0} \approx \mathrm{Chow}(X) \right)$$

Decomposition thm.:

$$\pi : X \to Y \text{ proper} \implies \pi_! : \mathrm{Pure}(X) \to \mathrm{Pure}(Y)$$

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & X \arrow[d, "f"] & U \arrow[l, "j^*"'] \arrow[d] \\ \{0\} \arrow[r] & \mathbb{C} & \mathbb{C}^\times \arrow[l] \end{tikzcd}$$

$\rightsquigarrow$

$$\begin{tikzcd} {}^p\psi^{un}_f(\mathcal{F}) \arrow[r, shift left, "\mathrm{can}"] & {}^p\varphi^{un}_f(\mathcal{F}) \arrow[l, shift left, "\mathrm{var}"] \end{tikzcd}$$

8 Advanced topics

⚠ Char $k = 0$

$$\mathrm{Pure}(X) = \langle \mathrm{IC} \rangle_{\oplus, [1]} \subset D^b(X) \quad \left(\mathrm{Pure} \approx D^b_{mix}(X)^{w=0} \approx \mathrm{Chow}(X) \right)$$

Decomposition thm.:

$\pi : X \to Y$ proper $\Longrightarrow$ $\pi_! : \mathrm{Pure}(X) \to \mathrm{Pure}(Y)$

$$\begin{tikzcd}
Z \arrow[r] \arrow[d] & X \arrow[d, "f"] & U \arrow[l, "j^*"'] \arrow[d] \\
\{0\} \arrow[r] & \mathbb{C} & \mathbb{C}^\times \arrow[l]
\end{tikzcd}$$

$\rightsquigarrow$

$$\begin{tikzcd}
{}^p\psi^{un}_f(\mathcal{F}) \arrow[r, shift left, "can"] & {}^p\varphi^{un}_f(\mathcal{F}) \arrow[l, shift left, "var"] \\
\text{nearby cycles} \arrow[u] & \text{Vanishing cycles} \arrow[u]
\end{tikzcd}$$

8 Advanced topics

⚠ char $k = 0$

$\mathrm{Pure}(X) = \langle \mathrm{IC} \rangle_{\oplus, [1]} \subset D^b(X)$ $\left(\mathrm{Pure} \approx D^b_{mix}(X)^{w=0} \approx \mathrm{Chow}(X) \right)$

Decomposition thm.:

$\pi : X \to Y$ proper $\Longrightarrow$ $\pi_! : \mathrm{Pure}(X) \to \mathrm{Pure}(Y)$

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & X \arrow[d, "f"] & U \arrow[l, "j^*"'] \arrow[d] \\ \{0\} \arrow[r] & \mathbb{C} & \mathbb{C}^\times \arrow[l] \end{tikzcd}$$

$\rightsquigarrow$

$$\begin{tikzcd} {}^p\psi^{un}_f(\mathcal{F}) \arrow[r, shift left, "can"] & {}^p\varphi^{un}_f(\mathcal{F}) \arrow[l, shift left, "var"] \end{tikzcd}$$

↑ nearby cycles $\approx$ coh. of Milnor fibre

↑ vanishing cycles

8 Advanced topics

⚠ Char $k=0$

$$\mathrm{Pure}(X) = \langle IC \rangle_{\oplus,[1]} \subset D^b(X) \quad \left(\mathrm{Pure} \approx D^b_{mix}(X)^{w=0} \approx \mathrm{Chow}(X) \right)$$

Decomposition thm.:

$\pi: X \to Y$ proper $\Rightarrow$ $\pi_! : \mathrm{Pure}(X) \to \mathrm{Pure}(Y)$

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & X \arrow[d, "f"] & U \arrow[l, "j^*"'] \arrow[d] \\ \{0\} \arrow[r] & \mathbb{C} & \mathbb{C}^\times \arrow[l] \end{tikzcd} \rightsquigarrow \begin{tikzcd} {}^p\psi^{un}_f(\mathcal{F}) \arrow[r, shift left, "can"] & {}^p\varphi^{un}_f(\mathcal{F}) \arrow[l, shift left, "var"] \end{tikzcd}$$

nearby cycles
$\approx$ coh. of Milnor fibre

vanishing cycles

Thm:

$\mathcal{F} \in \mathrm{Perv}(X)$

8 Advanced topics

⚠ Char $k = 0$

$\mathrm{Pure}(X) = \langle IC \rangle_{\oplus, [1]} \subset D^b(X)$ $\quad$ $\left(\mathrm{Pure} \approx D^b_{mix}(X)^{w=0} \approx \mathrm{Chow}(X) \right)$

Decomposition thm.:

$\pi : X \to Y$ proper $\implies$ $\pi_! : \mathrm{Pure}(X) \to \mathrm{Pure}(Y)$

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & X \arrow[d, "f"] & U \arrow[l, "j^*"'] \arrow[d] \\ \{0\} \arrow[r] & \mathbb{C} & \mathbb{C}^\times \arrow[l] \end{tikzcd}$$

$\rightsquigarrow$

$$\begin{tikzcd} {}^p\psi^{un}_f(\mathcal{F}) \arrow[r, shift left, "can"] & {}^p\varphi^{un}_f(\mathcal{F}) \arrow[l, shift left, "var"] \end{tikzcd}$$

$\uparrow$ nearby cycles $\approx$ coh. of Milnor fibre

$\uparrow$ Vanishing cycles

Thm:

$\mathcal{F} \in \mathrm{Perv}(X)$ det. by

8 Advanced topics

⚠ char $k = 0$

$$\mathrm{Pure}(X) = \langle IC \rangle_{\oplus, [1]} \subset D^b(X) \quad \left(\mathrm{Pure} \approx D^b_{mix}(X)^{w=0} \approx \mathrm{Chow}(X) \right)$$

Decomposition thm.:

$$\pi : X \to Y \text{ proper} \quad \Longrightarrow \quad \pi_! : \mathrm{Pure}(X) \to \mathrm{Pure}(Y)$$

$$\begin{tikzcd}
Z \arrow[r] \arrow[d] & X \arrow[d, "f"] & U \arrow[l, "j^*"'] \arrow[d] \\
\{0\} \arrow[r] & \mathbb{C} & \mathbb{C}^\times \arrow[l]
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
{}^p\psi^{un}_f(\mathcal{F}) \arrow[r, shift left, "can"] & {}^p\varphi^{un}_f(\mathcal{F}) \arrow[l, shift left, "var"]
\end{tikzcd}$$

nearby cycles

$\approx$ coh. of Milnor fibre

vanishing cycles

Thm:

$$\mathcal{F} \in \mathrm{Perv}(X) \text{ det. by } \Big(\underset{\substack{\cap \\ \mathrm{Perv}(U)}}{j^*(\mathcal{F})}, \ \underset{\substack{\cap \\ \mathrm{Perv}(Z)}}{{}^p\varphi^{un}_f(\mathcal{F})}, \ can, \ var \Big)$$

II. Flag varieties

1. Finite case

$$\overset{\substack{\text{max.}\\ \text{torus}\\ \downarrow}}{T} \subset \overset{\substack{\text{Borel}\\ \downarrow}}{B} \subset \overset{\substack{\text{split}\\ \text{reductive group}\\ \downarrow}}{\mathcal{G}}$$

1. Finite case

$$\begin{tikzcd}[column sep=small] \substack{\text{max.}\\ \text{torus}} \arrow[d] & & \text{Borel} \arrow[d] & & \substack{\text{split}\\ \text{reductive group}} \arrow[d] \\ T \arrow[r, phantom, "\subset"] & {} & B \arrow[r, phantom, "\subset"] & {} & G \end{tikzcd} \rightsquigarrow \begin{tikzcd} \text{simple coroots} \arrow[d] & \text{cocharacter lattice} \arrow[d] \\ I^\vee \arrow[r, phantom, "\subset"] \arrow[d, leftrightarrow] & X_*(T) \arrow[d, leftrightarrow, "{\langle , \rangle}"] \\ I \arrow[r, phantom, "\subset"] & X^*(T) \\ \text{simple roots} \arrow[u] & \text{character lattice} \arrow[u] \end{tikzcd}$$

1. Finite case

$$\overset{\substack{\text{max.}\\ \text{torus}\\ \downarrow}}{T} \subset \overset{\substack{\text{Borel}\\ \downarrow}}{B} \subset \overset{\substack{\text{split}\\ \text{reductive group}\\ \downarrow}}{G} \rightsquigarrow$$

$$\begin{tikzcd}
\overset{\substack{\text{simple coroots}\\ \downarrow}}{I^\vee} \arrow[d, leftrightarrow] & \subset & \overset{\substack{\text{cocharacter lattice}\\ \downarrow}}{X_*(T)} \arrow[d, leftrightarrow, "\langle , \rangle"] \\
\underset{\substack{\uparrow\\ \text{simple roots}}}{I} & \subset & \underset{\substack{\uparrow\\ \text{character lattice}}}{X^*(T)}
\end{tikzcd}$$

$$\rightsquigarrow \quad W(R) = \overset{\substack{\text{Weyl group}\\ \downarrow}}{W} = N_G(T)/T$$

$$\begin{gathered} \| \\ \langle s_{\alpha_i} \mid i \in I \rangle \end{gathered}$$

1. Finite case

$$\overset{\substack{\text{max.}\\ \text{torus}}}{\overset{\downarrow}{T}} \subset \overset{\text{Borel}}{\overset{\downarrow}{B}} \subset \overset{\substack{\text{split}\\ \text{reductive group}}}{\overset{\downarrow}{G}} \quad \rightsquigarrow$$

$$\begin{tikzcd}
\overset{\text{simple coroots}}{\overset{\downarrow}{I^\vee}} \arrow[r, phantom, "\subset"] \arrow[d, leftrightarrow] & \overset{\text{cocharacter lattice}}{\overset{\downarrow}{X_*(T)}} \arrow[d, leftrightarrow, "{\langle , \rangle}"] \\
\underset{\text{simple roots}}{\underset{\uparrow}{I}} \arrow[r, phantom, "\subset"] & \underset{\text{character lattice}}{\underset{\uparrow}{X^*(T)}}
\end{tikzcd}$$

$$\rightsquigarrow \quad W(R) = \overset{\substack{\text{Weyl group}\\ \downarrow}}{W} = N_G(T)/T$$

$$W = \langle s_{\alpha_i} \mid i \in I \rangle$$

$$B \subset P, Q \subset G$$

1. Finite case

$$\overset{\substack{\text{max.}\\ \text{torus}\\ \downarrow}}{T} \subset \overset{\substack{\text{Borel}\\ \downarrow}}{B} \subset \overset{\substack{\text{split}\\ \text{reductive group}\\ \downarrow}}{G} \rightsquigarrow$$

$$\begin{tikzcd}
\overset{\substack{\text{simple coroots}\\ \downarrow}}{I^\vee} \arrow[r, phantom, "\subset"] \arrow[d, leftrightarrow] & \overset{\substack{\text{cocharacter lattice}\\ \downarrow}}{X_*(T)} \arrow[d, leftrightarrow, "{\langle , \rangle}"] \\
\underset{\substack{\uparrow\\ \text{simple roots}}}{I} \arrow[r, phantom, "\subset"] & \underset{\substack{\uparrow\\ \text{character lattice}}}{X^*(T)}
\end{tikzcd}$$

$$\rightsquigarrow \quad W(R) = \overset{\substack{\text{Weyl group}\\ \downarrow}}{W} = N_G(T)/T$$

$$\|$$

$$\langle s_{\alpha_i} \mid i \in I \rangle$$

$$B \subset P, Q \subset G \rightsquigarrow I_P, I_Q \subset I$$

1. Finite case

$$\overset{\substack{\text{max.}\\ \text{torus}}}{T} \subset \overset{\text{Borel}}{B} \subset \overset{\substack{\text{split}\\ \text{reductive group}}}{G} \quad \rightsquigarrow$$

simple coroots cocharacter lattice

$$\begin{tikzcd}
I^\vee \arrow[d, leftrightarrow] \arrow[r, phantom, "\subset"] & X_*(T) \arrow[d, leftrightarrow, "{\langle , \rangle}"] \\
I \arrow[r, phantom, "\subset"] & X^*(T)
\end{tikzcd}$$

simple roots character lattice

$$\rightsquigarrow \quad W(R) = \overset{\text{Weyl group}}{W} = N_G(T)/T$$

$$\| $$

$$\langle s_{\alpha_i} \mid i \in I \rangle$$

$$B \subset P, Q \subset G \quad \rightsquigarrow \quad I_P, I_Q \subset I \quad \rightsquigarrow \quad W_P, W_Q \subset W$$

$$W_P = \langle s_{\alpha_i} \mid i \in I_P \rangle \qquad W_Q = \langle s_{\alpha_i} \mid i \in I_Q \rangle$$

1. Finite case

$$\overset{\text{max. torus}}{\overset{\downarrow}{T}} \subset \overset{\text{Borel}}{\overset{\downarrow}{B}} \subset \overset{\text{split reductive group}}{\overset{\downarrow}{G}} \rightsquigarrow \begin{tikzcd} \overset{\text{simple coroots}}{\overset{\downarrow}{I^\vee}} \arrow[d, leftrightarrow] & \subset \overset{\text{cocharacter lattice}}{\overset{\downarrow}{X_*}}(T) \arrow[d, leftrightarrow, "{\langle , \rangle}"] \\ \underset{\text{simple roots}}{\underset{\uparrow}{I}} & \subset \underset{\text{character lattice}}{\underset{\uparrow}{X^*}}(T) \end{tikzcd} \rightsquigarrow W(R) = \overset{\text{Weyl group}}{\overset{\downarrow}{W}} = N_G(T)/T$$

$$W = \langle s_{\alpha_i} \mid i \in I \rangle$$

$$B \subset P, Q \subset G \rightsquigarrow I_P, I_Q \subset I \rightsquigarrow W_P, W_Q \subset W$$

$$W_P = \langle s_{\alpha_i} \mid i \in I_P \rangle \quad W_Q = \langle s_{\alpha_i} \mid i \in I_Q \rangle$$

$$G = \overset{\text{Bruhat decomposition}}{\overset{\downarrow}{\boxplus}}_{w \in W_Q \backslash W / W_P} Q \dot{w} P$$

$Q = B$, $\operatorname{char} k = 0$

$Q = B$, char $k = 0$

$$\underset{\substack{\uparrow \\ \text{(partial) flag variety}}}{G/P}$$

$Q = B$, char $k = 0$

Bruhat stratification

$$G/P = \biguplus_{w \in W/W_P} B\dot{w}P/P$$

(partial) flag variety

$Q = B$, char $k = 0$

Bruhat stratification

$$G/P = \biguplus_{w \in W/W_P} B\dot{w}P/P$$

Bruhat cell $X_w \cong \mathbb{C}^{\ell(w)}$

(partial) flag variety

$Q = B$, $\operatorname{char} k = 0$

Bruhat stratification

$$\begin{tikzcd}
G/P = \biguplus_{w \in W/W_P} B\dot{w}P/P \arrow[d, squiggly] \\
\operatorname{Perv}_{(B)}(G/P) \subset D^b_{(B)}(G/P)
\end{tikzcd}$$

(partial) flag variety

Bruhat cell $X_w \cong \mathbb{C}^{\ell(w)}$

$Q = B$, char $k = 0$

Bruhat stratification

$$G/P = \biguplus_{w \in W/W_P} B\dot{w}P/P$$

(partial) flag variety

Bruhat cell $X_w \cong \mathbb{C}^{\ell(w)}$

$$\rightsquigarrow$$

Schubert variety

$$\mathrm{Perv}_{(B)}(G/P) \subset D^b_{(B)}(G/P) \supset \mathrm{Pure}_{(B)}(G/P) = \langle IC(\overline{X}_w) \rangle_{\oplus, [1]}$$

$\mathcal{K}$, $\mathrm{Par}_{(B)}(G/P)$

$Q, P = B$, char $k = 0$

$$h : \mathrm{Obj}\left(D^b_{(B)}(G/B)\right) \longrightarrow \mathcal{H} = \sum_{w \in W} \mathbb{Z}[v, v^{-1}] \overset{\substack{\text{standard basis} \\ \downarrow}}{h_w}$$

$Q, P = B$, char $k = 0$

$$h : \mathrm{Obj}\left(D^b_{(B)}(G/B)\right) \longrightarrow \mathcal{H} = \sum_{w\in W} \mathbb{Z}[v,v^{-1}]\, \overset{\substack{\text{standard basis}\\ \downarrow}}{h_w}$$

$$\mathcal{F} \longmapsto \sum_{w\in W} \sum_{i\in\mathbb{Z}} \dim_k H^i(\mathcal{F}|_{X_w})\, v^{\ell(w)-i}\, h_w$$

$Q, P = B, \operatorname{char} k = 0$

⚠ Does not descend to K_0

standard basis

$$h : \operatorname{Obj}(D^b_{(B)}(G/B)) \longrightarrow \mathcal{H} = \sum_{w \in W} \mathbb{Z}[v, v^{-1}] h_w$$

$$\mathcal{F} \longmapsto \sum_{w \in W} \sum_{i \in \mathbb{Z}} \dim_k H^i(\mathcal{F}|_{X_w}) v^{\ell(w)-i} h_w$$

$Q, P = B$, char $k = 0$

⚠ Does not descend to K_0

standard basis

$$\begin{aligned} h : \mathrm{Obj}(D^b_{(B)}(G/B)) &\longrightarrow \mathcal{H} = \sum_{w \in W} \mathbb{Z}[v, v^{-1}] h_w \\ \mathcal{F} &\longmapsto \sum_{w \in W} \sum_{i \in \mathbb{Z}} \dim_k H^i(\mathcal{F}|_{X_w}) v^{\ell(w)-i} h_w \end{aligned}$$

<u>Thm</u> (Kazdhan-Lusztig)

$$IC(\overline{X}_w, k) \xmapsto{h}$$

Q, $P = B$, char $k = 0$

⚠ Does not descend to K_0

standard basis

$$h : \mathrm{Obj}(D^b_{(B)}(G/B)) \longrightarrow \mathcal{H} = \sum_{w \in W} \mathbb{Z}[v, v^{-1}] h_w$$

$$\mathcal{F} \longmapsto \sum_{w \in W} \sum_{i \in \mathbb{Z}} \dim_k H^i(\mathcal{F}|_{X_w}) v^{\ell(w)-i} h_w$$

<u>Thm</u> (Kazdhan-Lusztig)

Kazhdan-Lusztig basis

$$IC(\overline{X}_w, k) \xmapsto{h} \underline{h}_w$$

Lemma: $\mathcal{F} \in \mathrm{Pure}_{(B)}(\mathcal{G}/B)$

(a)

(b)

Lemma: $\mathcal{F} \in \mathrm{Pure}_{(B)}(G/B)$

(a) $\pi_s : G/B \to G/P_s$ $\qquad P_s = \overline{BsB}$ $\qquad s \in S$

(b)

Lemma: $\mathcal{F} \in \mathrm{Pure}_{(B)}(G/B)$

(a) $\pi_s : G/B \to G/P_s \qquad P_s = \overline{BsB} \qquad s \in S$

$\pi_{s!}\pi_s^* \mathcal{F}[1]$

Lemma: $\mathcal{F} \in \mathrm{Pure}_{(B)}(\mathcal{G}/B)$

(a) $\pi_s : \mathcal{G}/B \to \mathcal{G}/P_s \qquad P_s = \overline{BsB}$

$h(\pi_{s!}\pi_s^* \mathcal{F}[1])$

Lemma: $\mathcal{F} \in \mathrm{Pure}_{(B)}(G/B)$

(a) $\pi_s : G/B \to G/P_s \qquad P_s = \overline{BsB} \qquad s \in S$

$h(\pi_{s!}\pi_s^* \mathcal{F}[1]) = h(\mathcal{F})\,\underline{h}_s$

Lemma: $\mathcal{F} \in \mathrm{Pure}_{(B)}(G/B)$

(a) $\pi_s : G/B \to G/P_s$ $\qquad P_s = \overline{BsB} \qquad s \in S$

$$h(\pi_{s!}\pi_s^* \mathcal{F}[1]) = h(\mathcal{F})\,\underline{h}_s$$

$$= h(IC(\overset{\mathbb{P}^1}{\overset{\|}{\overline{X_s}}}))$$

$$= h_s + v$$

Lemma: $\mathcal{F} \in \mathrm{Pure}_{(B)}(G/B)$

(a) $\pi_s : G/B \to G/P_s \qquad P_s = \overline{BsB} \qquad s \in S$

$$h(\pi_{s!}\pi_s^*\mathcal{F}[1]) = h(\mathcal{F})\,\underline{h}_s$$

π_s^*: $= \mathcal{F} * k_{\underset{\underset{D_{B\times B}(G)}{\pitchfork}}{\overline{BsB}}}[1]$

$\underline{h}_s$: $= h(IC(\overset{\overset{\mathbb{P}^1}{\shortparallel}}{\overline{X}_s}))$

$= h_s + v$

$$\left.\begin{tikzcd}
X_{ws} \arrow[drr] & & \\
\cup X_w \arrow[rr, "\sim"] \arrow[d] \arrow[drr, phantom, "\lrcorner"] & & X_w \arrow[d] \\
G/B \arrow[rr, "\mathbb{P}^1\text{- bundle}"'] & & G/P_s
\end{tikzcd}\right] \leftsquigarrow$$

+ Parity vanishing

Lemma: $\mathcal{F} \in \mathrm{Pure}_{(B)}(G/B)$

(a) $\pi_s : G/B \to G/P_s \qquad P_s = \overline{BsB} \qquad s \in S$

$$h(\pi_{s!}\pi_s^* \mathcal{F}[1]) = h(\mathcal{F})\,\underline{h}_s$$

$\pi_{s!}\pi_s^*\mathcal{F}[1] = \mathcal{F} * \underset{\substack{\cap \\ D_{B\times B}(G)}}{k_{\overline{BsB}}[1]}$

$\underline{h}_s = h(IC(\overset{\overset{\mathbb{P}^1}{\shortparallel}}{\overline{X}_s})) = h_s + v$

$$\begin{tikzcd}
X_{ws} \arrow[rd] & \\
\sqcup X_w \arrow[r, "\sim"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X_w \arrow[d] \\
G/B \arrow[r, "\mathbb{P}^1\text{- bundle}"'] & G/P_s
\end{tikzcd}$$

+ parity vanishing

(b) $h(\mathbb{D}\mathcal{F}) = \iota h(\mathcal{F})$

Lemma: $\mathcal{F} \in \mathrm{Pure}_{(B)}(G/B)$

(a) $\pi_s : G/B \to G/P_s \qquad P_s = \overline{BsB} \qquad s \in S$

$$h(\pi_{s!}\pi_s^* \mathcal{F}[1]) = h(\mathcal{F})\,\underline{h}_s$$

$\pi_{s!}\pi_s^*\mathcal{F}[1] = \mathcal{F} * \underset{\substack{\cap \\ D_{B\times B}(G)}}{k_{\overline{BsB}}}[1]$

$\underline{h}_s = h(IC(\overset{\mathbb{P}^1}{\overset{\parallel}{\overline{X}_s}}))$

$= h_s + v$

$$\begin{tikzcd}
X_{ws} \arrow[drr] & & \\
\cup X_w \arrow[rr, "\sim"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & & X_w \arrow[d] \\
G/B \arrow[rr, swap, "\mathbb{P}^1\text{- bundle}"] & & G/P_s
\end{tikzcd}$$

+ Parity vanishing

(b) $h(\mathbb{D}\mathcal{F}) = \iota\, h(\mathcal{F})$

$\iota(v) = v^{-1} \quad \iota(h_x) = h_{x^{-1}}^{-1}$

"Bar involution"

Lemma: $\mathcal{F} \in \mathrm{Pure}_{(B)}(\mathcal{G}/B)$

(a) $\pi_s : \mathcal{G}/B \to \mathcal{G}/P_s$ $\qquad P_s = \overline{BsB}$ $\qquad s \in S$

$$h(\pi_{s!}\pi_s^* \mathcal{F}[1]) = h(\mathcal{F})\,\underline{h}_s$$

$\pi_{s!}\pi_s^*$ $= \mathcal{F} * k_{\overline{BsB}}[1]$, $\overline{BsB} \in D_{B\times B}(\mathcal{G})$

$\underline{h}_s$ $= h(IC(\overset{\mathbb{P}^1}{\overset{\shortparallel}{\overline{X}_s}})) = h_s + v$

$$\begin{tikzcd} X_{ws} \arrow[rr] & & X_w \\ \subset X_w \arrow[rr, "\sim"] \arrow[d] & \lrcorner & X_w \arrow[d] \\ \mathcal{G}/B \arrow[rr] & & \mathcal{G}/P_s \end{tikzcd}$$

$\mathbb{P}^1$- bundle

\+ parity vanishing

(b) $h(\mathbb{D}\mathcal{F}) = \iota\, h(\mathcal{F})$

$\iota(v) = v^{-1}$ $\quad \iota(h_x) = h^{-1}_{x^{-1}}$

"Bar involution"

$\mathrm{Pure}_{(B)}(\mathcal{G}/B) = \langle i_{e!}k \rangle_{\pi_s^*\pi_{s!},\, \oplus,\, \mathbb{C}}$

Proof (Thm.)

(a) $\mathbb{D}(IC(\overline{X}_w, k)) = IC(\overline{X}_w, k)$

Proof (Thm.)

(a) $\mathbb{D}(IC(\overline{X}_w, k)) = IC(\overline{X}_w, k) \Rightarrow \mathrm{ch}(IC(\overline{X}_w, k)) = h(IC(\overline{X}_w, k))$

Proof (Thm.)

(a) $\mathbb{D}(IC(\overline{X}_w, k)) = IC(\overline{X}_w, k) \Rightarrow \iota h(IC(\overline{X}_w, k)) = h(IC(\overline{X}_w, k))$

(b) $h(IC(\overline{X}_w, k)) = h_w + \sum_{x < w} v\mathbb{Z}[v] h_x$

General fact about IC-sheaves

Proof (Thm.)

(a) $\mathbb{D}(IC(\overline{X}_w, k)) = IC(\overline{X}_w, k) \Rightarrow h(IC(\overline{X}_w, k)) = h(IC(\overline{X}_w, k))$

(b) $h(IC(\overline{X}_w, k)) = h_w + \sum_{x < w} v\mathbb{Z}[v] h_x$

↑ General fact about IC-sheaves

$\Rightarrow h(IC(\overline{X}_w, k))$ fulfills universal property of $\underline{h}_w$

Proof (Thm.)

(a) $\mathbb{D}(IC(\overline{X}_w, k)) = IC(\overline{X}_w, k) \Rightarrow h(IC(\overline{X}_w, k)) = h(IC(\overline{X}_w, k))$

(b) $h(IC(\overline{X}_w, k)) = h_w + \sum\limits_{x<w} v\mathbb{Z}[v] h_x$

$\uparrow$ General fact about IC-sheaves

$\Rightarrow$ $h(IC(\overline{X}_w, k))$ fulfills universal property of $\underline{h}_w$

$$\mathcal{H} \hookrightarrow K_0^{\oplus}(\mathrm{Pure}_{(B)}(g/B))$$

Proof (Thm.)

(a) $\mathbb{D}(IC(\overline{X}_w, k)) = IC(\overline{X}_w, k) \Rightarrow \iota h(IC(\overline{X}_w, k)) = h(IC(\overline{X}_w, k))$

(b) $h(IC(\overline{X}_w, k)) = h_w + \sum_{x<w} v\mathbb{Z}[v] h_x$

$\uparrow$ General fact about IC-sheaves

$\Rightarrow h(IC(\overline{X}_w, k))$ fulfills universal property of $\underline{h}_w$

$$\mathcal{H} \xrightarrow{\sim} K_0^{\oplus}(\mathrm{Pure}_{(B)}(G/B))$$

$$\underline{h}_s = K_0^{\oplus}(\pi_s^* \pi_{s!}[1])$$

Proof (Thm.)

(a) $\mathbb{D}(IC(\overline{X}_w, k)) = IC(\overline{X}_w, k) \Rightarrow c\,h(IC(\overline{X}_w, k)) = h(IC(\overline{X}_w, k))$

(b) $h(IC(\overline{X}_w, k)) = h_w + \sum_{x<w} v\mathbb{Z}[v] h_x$

↑ General fact about IC-sheaves

$\Rightarrow$ $h(IC(\overline{X}_w, k))$ fulfills universal property of $\underline{h}_w$

$$\mathcal{H} \hookrightarrow K_0^{\oplus}(\mathrm{Pure}_{(B)}(G/B))$$

$\underline{h}_s = K_0^{\oplus}(\pi_s^* \pi_{s!} [1])$

↑

regular module

Proof (Thm.)

(a) $\mathbb{D}(IC(\overline{X}_w, k)) = IC(\overline{X}_w, k) \Rightarrow \iota h(IC(\overline{X}_w, k)) = h(IC(\overline{X}_w, k))$

(b) $h(IC(\overline{X}_w, k)) = h_w + \sum_{x<w} v\mathbb{Z}[v] h_x$

↑ General fact about IC-sheaves

$\Rightarrow h(IC(\overline{X}_w, k))$ fulfills universal property of $\underline{h}_w$

$$\mathcal{H} \xrightarrow{\sim} K_0^{\oplus}(\mathrm{Pure}_{(B)}(G/B)) \left(= K_0(D^{b,mix}_{(B)}(G/B))\right)$$

$\underline{h}_s = K_0^{\oplus}(\pi_s^* \pi_{s!}[1])$

↑

regular module

2. Kac-Moody

$$K = \underset{\underset{\displaystyle\sum_{i=-n}^{\infty} a_i t^i}{\uparrow}}{\mathbb{C}((t))} \supset \underset{\underset{\displaystyle\sum_{i=0}^{\infty} a_i t^i}{\uparrow}}{\mathbb{C}[[t]]} = \mathcal{O}$$

2. Kac-Moody

$$K = \underset{\underset{\textstyle\sum_{i=-n}^{\infty} a_i t^i}{\uparrow}}{\mathbb{C}((t))} \supset \underset{\underset{\textstyle\sum_{i=0}^{\infty} a_i t^i}{\uparrow}}{\mathbb{C}[[t]]} = \mathcal{O}$$

$\{$split reductive$\} \subset \{$Kac-Moody groups$\}$

2. Kac-Moody

$$K = \underset{\underset{\sum_{i=-n}^{\infty} a_i t^i}{\uparrow}}{\mathbb{C}((t))} \supset \underset{\underset{\sum_{i=0}^{\infty} a_i t^i}{\uparrow}}{\mathbb{C}[[t]]} = \mathcal{O}$$

$$\{\text{split reductive}\} \subset \{\text{Kac-Moody groups}\} \supset \{\text{Loop groups}\}$$

2. Kac-Moody

$$\begin{tikzcd}[column sep=small] K = \mathbb{C}((t)) & \supset & \mathbb{C}[[t]] = \mathcal{O} \\ \sum_{i=-n}^{\infty} a_i t^i \arrow[u] & & \sum_{i=0}^{\infty} a_i t^i \arrow[u] \end{tikzcd}$$

$$\{\text{split reductive}\} \subset \{\text{Kac-Moody groups}\} \supset \{\text{Loop groups}\}$$

$$G(\mathbb{C}) \qquad\qquad\qquad G(K) = LG$$

2. Kac-Moody

$$K = \underset{\substack{\uparrow \\ \sum_{i=-n}^{\infty} a_i t^i}}{\mathbb{C}((t))} \supset \underset{\substack{\uparrow \\ \sum_{i=0}^{\infty} a_i t^i}}{\mathbb{C}[[t]]} = \mathcal{O}$$

$$\{\text{split reductive}\} \subset \{\text{Kac-Moody groups}\} \supset \{\text{Loop groups}\}$$

$$G(\mathbb{C}) \qquad\qquad\qquad G(K) = LG$$

$$W \longrightarrow \underset{\substack{\uparrow \\ \text{(extended) affine Weyl group}}}{W_{\text{aff}}} = W \ltimes X_*(T)$$

2. Kac-Moody

$$K = \underset{\substack{\uparrow \\ \sum_{i=-n}^{\infty} a_i t^i}}{\mathbb{C}((t))} \supset \underset{\substack{\uparrow \\ \sum_{i=0}^{\infty} a_i t^i}}{\mathbb{C}[[t]]} = \mathcal{O}$$

{split reductive} ⊂ {Kac-Moody groups} ⊃ {Loop groups}

$$G(\mathbb{C}) \qquad\qquad\qquad G(K) = LG$$

$$W \longrightarrow \underset{\substack{\uparrow \\ \text{(extended) affine Weyl group}}}{W_{aff}} = W \ltimes X_*(T)$$

$$\{ \underset{\substack{\uparrow \\ \text{Borel}}}{B} \subset \underset{\substack{\uparrow \\ \text{parabolic}}}{P} \subset G \} \qquad\qquad \{ \underset{\substack{\uparrow \\ \text{Iwahori}}}{I} \subset \underset{\substack{\uparrow \\ \text{parahoric}}}{Q} \subset LG \}$$

2. Kac-Moody

$$\underset{\underset{\sum\limits_{i=-n}^{\infty} a_i t^i}{\uparrow}}{K = \mathbb{C}((t))} \supset \underset{\underset{\sum\limits_{i=0}^{\infty} a_i t^i}{\uparrow}}{\mathbb{C}[[t]]} = \mathcal{O}$$

{split reductive} ⊂ {Kac-Moody groups} ⊃ {Loop groups}

$$\begin{tikzcd} G(\mathbb{C}) & G(K) = LG \\ W \arrow[r] & W_{\text{aff}} = W \ltimes X_*(T) \end{tikzcd}$$

$$\underset{\underset{\text{(extended) affine Weyl group}}{\uparrow}}{W_{\text{aff}}}$$

$$\{ \underset{\underset{\text{Borel}}{\uparrow}}{B} \subset \underset{\underset{\text{parabolic}}{\uparrow}}{P} \subset G \} \qquad \{ \underset{\underset{\text{Iwahori}}{\uparrow}}{I} \subset \underset{\underset{\text{parahoric}}{\uparrow}}{Q} \subset LG \}$$

f.e. • $Q = \underset{\underset{\text{positive loop group}}{\uparrow}}{L^+G} = G(\mathcal{O})$

2. Kac - Moody

$$K = \underset{\underset{\sum_{i=-n}^{\infty} a_i t^i}{\uparrow}}{\mathbb{C}((t))} \supset \underset{\underset{\sum_{i=0}^{\infty} a_i t^i}{\uparrow}}{\mathbb{C}[[t]]} = \mathcal{O}$$

{split reductive} ⊂ {Kac- Moody groups} ⊃ {Loop groups}

$$\mathcal{G}(\mathbb{C}) \qquad\qquad\qquad \mathcal{G}(K) = L\mathcal{G}$$

$$W \longrightarrow \underset{\underset{\text{(extended) affine Weyl group}}{\uparrow}}{W_{\text{aff}}} = W \ltimes X_*(T)$$

$$\{\underset{\underset{\text{Borel}}{\uparrow}}{B} \subset \underset{\underset{\text{parabolic}}{\uparrow}}{P} \subset \mathcal{G}\} \qquad\qquad \{\underset{\underset{\text{Iwahori}}{\uparrow}}{I} \subset \underset{\underset{\text{parahoric}}{\uparrow}}{Q} \subset L\mathcal{G}\}$$

i.e. • $Q = \underset{\underset{\text{positive loop group}}{\uparrow}}{L^+\mathcal{G}} = \mathcal{G}(\mathcal{O})$

•
$$\begin{tikzcd} I \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & L^+\mathcal{G} \arrow[d, "t=0"] \\ B \arrow[r] & \mathcal{G} \end{tikzcd}$$

affine flag variety

$$LG/I = \bigsqcup_{w \in W_{aff}} IwI/I$$

affine Grassmannian

$$LG/L^{+}G = \bigsqcup_{w \in W_{aff}/W = X_*(T)} IwLG^{+}/LG^{+}$$

$$= \bigsqcup_{w \in W\backslash W_{aff}/W = X_*^{+}(T)} LG^{+}wLG^{+}/LG^{+}$$

$\rightsquigarrow$ TBC

Thank you 💖