K-Motives and Koszul Duality
in
Geometric Representation Theory

I. Back to the 80's: How to come up with Koszul duality?

II. Status quo.

III. A K-theoretic perspective.

Jens Niklas Eberhardt, Uni Bonn

Koszul Duality: First Example

$$A = k[x]/x^2$$

$$\mathrm{End}_A(A) = A = k[x]/x^2$$

$$A^! = k[x]$$

$$\mathrm{Ext}^\bullet_{A^!}(k) = H^\bullet \mathrm{Hom}\left(\begin{array}{c} k[x] \\ \downarrow x \\ k[x] \end{array}, k \right) = k[x]/x^2$$

$$\begin{array}{ccc} A\text{-mod}^{\mathbb{Z}} & & D(A^!\text{-mod}) \\ \cup & & \cup \\ \langle A\langle i \rangle \mid i \in \mathbb{Z} \rangle_{\oplus} & \cong & \langle k[i] \mid i \in \mathbb{Z} \rangle_{\oplus} \end{array}$$

$$\begin{array}{ccc} D^b(A\text{-mod}^{\mathbb{Z}}) & \cong & D^b(A^!\text{-mod}^{\mathbb{Z}}) \\ A\langle i \rangle & \leftrightarrow & k\langle i \rangle [i] \end{array}$$

Koszulity of $A^!$ $\Rightarrow$ Formality of $\mathrm{End}^\bullet\left(\begin{array}{c} k[x]\langle 1 \rangle \\ \downarrow x \\ k[x] \end{array} \right)$

Koszul Duality: Second Example

$$\mathcal{O}_0(\mathfrak{sl}_2(\mathbb{C})) = \langle M(0), M(s.0)\rangle_{\text{Serre}} \subset \mathfrak{g}\text{-mod}_{\mathfrak{h}\text{-ss.}}$$

$$P(0) = M(0) = \begin{matrix} L(0) \\ L(s.0) \end{matrix} \qquad P(s.0) = \begin{matrix} M(s.0) \\ M(0) \end{matrix} = \begin{matrix} L(s.0) \\ L(0) \\ L(s.0) \end{matrix}$$

$$C = \operatorname{End}(P(s.0)) = \mathbb{C}[x]/x^2$$

$$A = \operatorname{End}(P(0) \oplus P(s.0))$$

$$= \mathbb{C}[\bullet \underset{\iota}{\overset{\pi}{\rightleftarrows}} \bullet]/\pi\iota$$

$$D^b_{\mathcal{S}}(\mathbb{P}^1, \mathbb{C}) \qquad \mathbb{P}^1 \qquad \mathcal{S} = \{\bigcirc, \bullet\}$$

$$\mathcal{IC}_e = \underline{\mathbb{C}}_{\bullet} \qquad \mathcal{IC}_s = \underline{\mathbb{C}}_{\bigcirc}[1]$$

$$C = \operatorname{Ext}^\bullet(\mathcal{IC}_s) = H^\bullet(\mathbb{P}^1, \mathbb{C}) = \mathbb{C}[x]/x^2$$

$$A = \operatorname{Ext}^\bullet(\mathcal{IC}_e \oplus \mathcal{IC}_s)$$

$$= \mathbb{C}[\bullet \underset{\iota}{\overset{\pi}{\rightleftarrows}} \bullet]/\pi\iota$$

$$\mathcal{O}_0^{\mathbb{Z}} = A^{op}\text{-mod}^{\mathbb{Z}} \qquad\qquad D^b_{\mathcal{S}}(\mathbb{P}^1, \mathbb{C})$$

$$\langle P(0), P(s.0)\rangle_{\langle 1\rangle, \oplus} \subset A^{op}\text{-mod}^{\mathbb{Z}} \qquad \cong \qquad \langle \mathcal{IC}_e, \mathcal{IC}_s\rangle_{[1], \oplus} \subset D^b_{\mathcal{S}}(\mathbb{P}^1, \mathbb{C})$$

Very mysterious!

Now what??

- Generalize? ✓
- Rep. th. meaning of $\mathcal{O}_0^{\mathbb{Z}}$? ✗
- Grading on $D^b_{\mathcal{S}}(\mathbb{P}^1, \mathbb{C})$? ✓
- Nat. functor? ✗

Koszul Duality: Generalize

[BG86] Alexander A. Beilinson and Victor Ginsburg, *Mixed categories, Ext-duality and representations (results and conjectures)*, Preprint, 1986.

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KATEGORIE $\mathcal{O}$, PERVERSE GARBEN UND MODULN ÜBER DEN KOINVARIANTEN ZUR WEYLGRUPPE

WOLFGANG SOERGEL

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KOSZUL DUALITY PATTERNS IN REPRESENTATION THEORY

ALEXANDER BEILINSON, VICTOR GINZBURG, AND WOLFGANG SOERGEL

$\mathfrak{h} \subset \mathfrak{b} \subset \mathfrak{g}$ complex reductive Lie algebra

$P(w.0) \in \mathcal{O}_0(\mathfrak{g}),\ w \in W$

$C = \mathrm{End}(P(w_0.0))$

$W =$ Weyl group

$C = S(\mathfrak{h})/S(\mathfrak{h})^W_+$

$T^\vee \subset B^\vee \subset G^\vee$ Langlands dual red. alg. grp.

$\mathcal{X}^\vee = G^\vee/B^\vee \quad \mathcal{X}^\vee_w = B^\vee w B^\vee/B^\vee \quad \mathcal{S} = \{\mathcal{X}^\vee_w \mid w \in W\}$

$\mathcal{IC}_w = \mathcal{IC}(\overline{\mathcal{X}^\vee_w}, \mathbb{C}) \in D^b_{\mathcal{S}}(\mathcal{X}^\vee, \mathbb{C})$

$C = \mathrm{Ext}^\bullet(\mathcal{IC}_{w_0}) = H^\bullet(\mathcal{X}^\vee, \mathbb{C})$

$$\begin{array}{ccccccc} & & C\text{-Mod} & & C\text{-Mod}^{\mathbb{Z}} & & \\ & & \cup & & \cup & & \\ \langle P(w.0)\rangle_\oplus & \xrightarrow[\sim]{\mathbb{V}} & \langle D_w\rangle_\oplus & \leftarrow \text{Soergel modules} \rightarrow & \langle D_w\rangle_{\oplus,\langle 1\rangle} & \xleftarrow[\sim]{\mathbb{H}} & \langle \mathcal{IC}_w \mid w \in W\rangle_{\oplus,[1]} \end{array}$$

$A = \mathrm{End}_C(\oplus D_w)$

$$\begin{array}{ccc} \mathcal{O}_0^{\mathbb{Z}} = A^!\text{-mod}^{\mathbb{Z}} & & D^b_{\mathcal{S}}(\mathcal{X}^\vee, \mathbb{C}) \\ \cup & & \cup \\ \langle P(w.0)\rangle_{\oplus,\langle 1\rangle} & \cong & \langle \mathcal{IC}_w\rangle_{\oplus,[1]} \end{array}$$

Koszul Duality: Mixed Geometry

How to obtain graded version $D_{\mathcal{G}}^{b,m}(\mathcal{X},\mathbb{C})$ of $D^b_{\mathcal{G}}(\mathcal{X}^\vee(\mathbb{C}),\mathbb{C})$?

(a) $D^b(\mathrm{MHM}(\mathcal{X}^\vee/\mathbb{C}))$ mixed Hodge modules

(b) $D^b_{mix}(\mathcal{X}^\vee/\mathbb{F}_p, \overline{\mathbb{Q}}_\ell)$ mixed ℓ-adic sheaves

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(c) $DM(\mathcal{X}^\vee/\mathbb{F}_p, \mathbb{Q})$ mixed motives

PERVERSE MOTIVES AND GRADED DERIVED CATEGORY $\mathcal{O}$

WOLFGANG SOERGEL AND MATTHIAS WENDT

$$\begin{array}{ccccccc}
D^b(\mathbb{Q}\text{-mod}^{\mathbb{Z}}) & \cong & \langle \mathbb{Q}(n)\rangle_\Delta & \subset & DM(\operatorname{Spec}\mathbb{F}_p,\mathbb{Q}) & & (n) \\
\downarrow & & \downarrow & & \downarrow \scriptstyle{v} & & \downarrow \\
D^b(\mathbb{Q}\text{-mod}) & \cong & \langle \mathbb{Q}\rangle_\Delta & \subset & D(pt,\mathbb{Q}) & & id
\end{array}$$

$\boxed{\text{Tate twist}}$ $\hat{=}$ shift of grading

$$\rightsquigarrow \quad \begin{array}{ccccc}
D_{\mathcal{G}}^{b,m}(\mathcal{X},\mathbb{Q}) & = & \langle \mathbb{Q}_{X_w^\vee}(n)\rangle_\Delta & \subset & DM(\mathcal{X}^\vee/\mathbb{F}_p,\mathbb{Q}) \\
\downarrow & & \downarrow & & \downarrow \scriptstyle{v} \\
D_{\mathcal{G}}^{b}(\mathcal{X}^\vee,\mathbb{Q}) & = & \langle \mathbb{Q}_{X_w^\vee}\rangle_\Delta & \subset & D(\mathcal{X}^\vee(\mathbb{C}),\mathbb{Q})
\end{array}$$

Amazing properties of $D^{b,m}_{\mathfrak{g}}(\mathcal{X}^\vee, \mathbb{Q})$:

- Six functors: $f^*, f_*, f_!, f^!, \otimes, \mathcal{H}om$
- Chow weight structure (= co-t-structure)

motives of smooth proj. vas. $/\mathcal{X}^\vee$

$$D^{b,m}_{\mathfrak{g}}(\mathcal{X}^\vee, \mathbb{Q})_{w=0} = \mathrm{Chow}(\mathcal{X}^\vee_{/\mathbb{F}_p}, \mathbb{Q}) \cap D^{b,m}_{\mathfrak{g}}(\mathcal{X}^\vee, \mathbb{Q})$$

decomp. thm.

$$= \langle \mathrm{I}\mathcal{C}_w(n)[2n] \rangle_{\oplus,\cong}$$

$$\cong \langle D_w \langle 2n \rangle \rangle_{\oplus} \subset C\text{-}\mathrm{Mod}^{\mathbb{Z}}$$

$$D^{b,m}_{\mathfrak{g}}(\mathcal{X}^\vee, \mathbb{Q}) \xrightarrow{\sim} K^b(D^{b,m}_{\mathfrak{g}}(\mathcal{X}^\vee, \mathbb{Q})_{w=0}) \cong K^b(\langle D_w \langle 2n \rangle \rangle_{\oplus})$$

Bondarko's weight complex functor

Koszul Duality: Status Quo

$$
\begin{tikzcd}
\overset{\langle 1 \rangle}{\overset{P(w.0)}{D^b(\mathcal{O}_0^{\mathbb{Z}}(\mathfrak{g}))}} \arrow[rr, leftrightarrow, "\sim"] & & \overset{(1)[2]}{\overset{IC_w}{D^{b,\mathrm{mix}}_{\mathfrak{g}}(\mathcal{Fl}^\vee, \mathbb{C})}} \arrow[d, "\wr"] \\
K^b(\mathrm{proj}(\mathcal{O}_0^{\mathbb{Z}}(\mathfrak{g}))) \arrow[u, "\wr"] \arrow[r, "\sim"] & K^b(\langle D_w \langle n \rangle \rangle_\oplus) & K^b(D^{b,\mathrm{mix}}_{\mathfrak{g}}(\mathcal{Fl}^\vee, \mathbb{C})_{w=0}) \arrow[l, "\sim"']
\end{tikzcd}
$$

Alternatively:

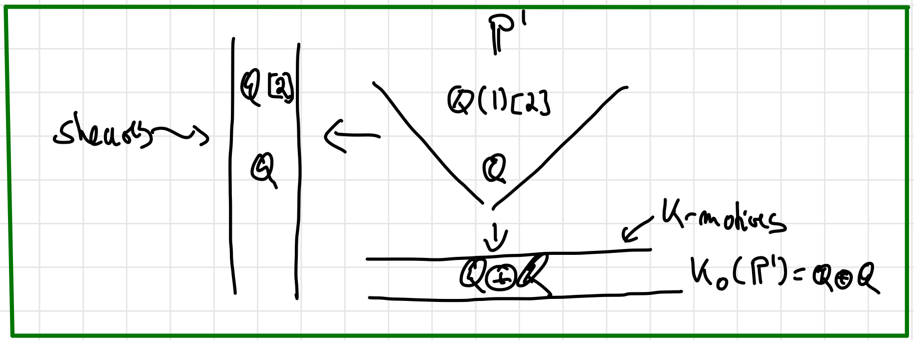
$$
\overset{(1)}{D^{b,\mathrm{mix}}_{\mathfrak{g}}(\mathcal{Fl}, \mathbb{Q})} \longleftrightarrow \overset{(1)[2]}{D^{b,\mathrm{mix}}_{\mathfrak{g}}(\mathcal{Fl}^\vee, \mathbb{Q})}
$$

Why not get rid of the grading??

$$\begin{tikzcd}
 & \langle 1 \rangle \arrow[rr, leftrightarrow] & & (1)[2] & \\
\langle 1 \rangle \arrow[d, mapsto] & D^b(\mathcal{O}_0^{\mathbb{Z}}(\mathfrak{g})) \arrow[rr, leftrightarrow] \arrow[d, "f"] & & D^{b,m}_{\mathfrak{g}}(\mathcal{F}\ell^\vee, \mathbb{C}) \arrow[d] & (1)[2] \arrow[d, mapsto] \\
\mathrm{id} & D^b(\mathcal{O}_0(\mathfrak{g})) \arrow[rr, leftrightarrow] & & ?? & \mathrm{id}
\end{tikzcd}$$

$$\begin{tikzcd} & DM(\mathcal{X}^\vee/\mathbb{F}_p, \mathbb{Q}) \arrow[dl, "(1) \mapsto \mathrm{id}"'] \arrow[dr, "(1)[2] \mapsto \mathrm{id}"] & \\ D(\mathcal{X}^\vee(\mathbb{C}), \mathbb{Q}) & & DK(\mathcal{X}^\vee/\mathbb{F}_p, \mathbb{Q}) \end{tikzcd}$$

commute with six fct.



K-Motives

"like sheaves but compute K-theory"

$$\mathrm{Hom}_{DK(S)}(\mathbb{Q}, \mathbb{Q}(p)[q]) = K_{2p-q}(S) \otimes \mathbb{Q}$$

vs $\qquad \Big\downarrow$ p-th Adams eigenspace

$$\mathrm{Hom}_{DM(S)}(\mathbb{Q}, \mathbb{Q}(p)[q]) = H^{p,q}(S, \mathbb{Q})$$

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NOTES ON MOTIVIC COHOMOLOGY

A. BEILINSON, R. MACPHERSON, AND V. SCHECHTMAN

Dedicated to Yu. I. Manin on the occasion of his 50th birthday

Amazing properties of $DK_{\mathcal{G}}(\mathcal{X}^\vee, \mathbb{Q}) = \langle \mathbb{Q}_{X^\vee_w} \rangle_\Delta \subset DK(\mathcal{X}^\vee, \mathbb{Q})$

- Six functors: $f^*, f_*, f_!, f^!, \otimes, \mathcal{H}om$
- Chow weight structure (= co-t-structure)

$$DK_{\mathcal{G}}(\mathcal{X}^\vee, \mathbb{Q})_{w=0} = \underbrace{KChow}_{\text{K-motives of smooth proj. vars.}/\mathcal{X}^\vee}(\mathcal{X}^\vee_{/\mathbb{F}_p}, \mathbb{Q}) \cap DK_{\mathcal{G}}(\mathcal{X}^\vee, \mathbb{Q})$$

$$= \langle I\mathcal{C}_w \rangle_{\oplus, \cong} \leftarrow \text{"intersection K-theory"}$$

$$\cong \langle D_w \rangle_\oplus \subset C\text{-Mod}$$

$$DK_{\mathcal{G}}(\mathcal{X}^\vee, \mathbb{Q}) \underset{\text{Bondarko's weight complex functor}}{\xrightarrow{\sim}} K^b(DK_{\mathcal{G}}(\mathcal{X}^\vee, \mathbb{Q})_{w=0}) \cong K^b(\langle D_w \rangle_\oplus)$$

"K-motives are Koszul-dual to constr. sheaves"

K-MOTIVES AND KOSZUL DUALITY

JENS NIKLAS EBERHARDT

$$\langle 1 \rangle \longleftrightarrow (1)[2]$$

$$\begin{tikzcd}
\langle 1 \rangle \arrow[d, mapsto] & D^b(\mathcal{O}_0^{\mathbb{Z}}(\mathfrak{g})) \arrow[r, leftrightarrow] \arrow[d, "v"] & D^{b,m}_{\mathcal{G}}(\mathcal{X}^\vee, \mathbb{C}) \arrow[d] & (1)[2] \arrow[d, mapsto] \\
\mathrm{id} & D^b(\mathcal{O}_0(\mathfrak{g})) \arrow[r, leftrightarrow] & DK_{\mathcal{G}}(\mathcal{X}^\vee, \mathbb{C}) & \mathrm{id}
\end{tikzcd}$$

Alternatively

$$D^b_{\mathcal{G}}(\mathcal{X}, \mathbb{Q}) \longleftrightarrow DK_{\mathcal{G}}(\mathcal{X}^\vee, \mathbb{Q})$$

Koszul Duality: Equivariant/Unipotenly Monodromic

EQUIVARIANT-CONSTRUCTIBLE KOSZUL DUALITY FOR DUAL TORIC VARIETIES

TOM BRADEN AND VALERY A. LUNTS

$\rightsquigarrow$

ON KOSZUL DUALITY FOR KAC-MOODY GROUPS

ROMAN BEZRUKAVNIKOV AND ZHIWEI YUN
WITH APPENDICES BY ZHIWEI YUN

Main Theorem. *There are equivalences of triangulated categories:*

- *Equivariant-monodromic duality (Theorem 5.2.1) which is a monoidal equivalence:*

$$\Phi : D^b_m(B^\vee\backslash G^\vee/B^\vee) \xrightarrow{\sim} \hat{D}^b_m(B \diagdown G \diagup B);$$

$$\begin{tikzcd}
H^*(BT^\vee)_{\mathbb{Q}} \arrow[r, hook] \arrow[d, equal] & H(BT^\vee)_{\mathbb{Q}} \arrow[d, equal] & K(BT^\vee)_{\mathbb{Q}} \arrow[l, "ch"', "\sim"] \arrow[d, equal] & K_{T^\vee}(*)_{\mathbb{Q}} = K(\mathrm{Rep}\,T^\vee)_{\mathbb{Q}} \arrow[l] \arrow[d, equal] = R_{T^\vee,\mathbb{Q}} = R \\
S(X(T^\vee)_{\mathbb{Q}}) & \prod S^i(X(T^\vee)_{\mathbb{Q}}) & \prod S^i(X(T^\vee)_{\mathbb{Q}}) & \mathbb{Q}[X(T^\vee)]
\end{tikzcd}$$

$$H(BT^\vee) = K(BT^\vee)_{\mathbb{Q}} \cong K_{T^\vee}(*)^{\wedge}_{\mathbb{Q},I}$$ Atiyah-Segal completion thm.

$$+ \quad \mathbb{Q}[\pi_1(T)] = \mathbb{Q}[X(T^\vee)] = K_{T^\vee}(*)_{\mathbb{Q}}$$ duality for tori

---

$$H(BT^\vee) = R^{\wedge}_{I} = \mathbb{Q}[\pi_1(T)]^{\wedge}_{I}$$

Why not get rid of grading + completion?

Conjecture: free monodromic tilting $\cap$ $\qquad$ equivariant intersection K-theory complex $\cap$

$$\begin{tikzcd}
D^b_{B\times B\text{-mon},f}(\mathcal{G}) \arrow[rr, leftrightarrow] \arrow[dr, "\wr"] & & DK_{B^\vee\times B^\vee, \mathcal{S}}(\mathcal{G}^\vee) \arrow[dl, "\sim"'] \\
& K^b(R\text{-}\mathrm{SBimod}) &
\end{tikzcd}$$

$\uparrow$

K-theoretic Soergel bimodules

$$\mathbb{Q}\pi_1(T) = R = K^T(pt)_{\mathbb{Q}}$$

Representation Theories from Coxeter Groups

M. J. DYER

8.2. Let A be the integral group ring $A = \mathbb{Z}[V]$ of V, with $\mathbb{Z}$-basis $\{e^\chi\}_{\chi\in V}$ and multiplication $e^\chi e^\psi = e^{\chi+\psi}$. The action of W on V induces an action of W as a group of ring automorphisms of A, with $we^\chi = e^{w(\chi)}$. For a finite standard parabolic subgroup W_K, let A^K denote the subalgebra of W_K-invariants on A. For $w \in X^K$, let $\sigma_w : A^K \to A$ denote the ring homomorphism $a \mapsto w^{-1}(a)$. The proofs of the following assertions are similar to those sketched previously in the case of the real reflection representation case, modulo facts from the invariant theory of A some of which can be found in [**26**], [**16**], [**18**].

THEOREM. *(a) The data $E = (A^K, \mathbb{Z}, A, X^K, \{\sigma_w\}_{w\in X^K})$ satisfies the conditions in 1.10. From the basic construction, one therefore has the categories $\mathcal{C}^K$, $\mathcal{P}^K$ and $\mathcal{E} = A^K$-mod where $\mathcal{A} = \mathrm{End}_{\mathcal{C}^K}(P)^{\mathrm{op}}$ for some projective generator $P = P^K$.*

Koszul Duality: Equivariant/Monodromic (Example)

$T = B = G$

$$
\begin{array}{c}
D^b_{T\times T\text{-mon}}(T) \\
\parallel \\
D_{T\text{-mon}}(T) \\
\parallel \\
D^b_t(T) \\
\parallel \\
D^b_{\pi_1(T)}(pt) \\
\parallel \\
D^b(R\text{-mod})
\end{array}
\qquad
\begin{array}{c}
\mathcal{L} \\
\big\downarrow \\
R
\end{array}
$$

$T^\vee = B^\vee = G^\vee$

$$
\begin{array}{c}
DK_{T\times T,s}(T) \\
\parallel \\
DK_{T,c}(pt) \\
\parallel \\
\langle \mathbb{Q} \rangle_\Delta \subset DK_T(pt) \\
\parallel \\
D^b(R\text{-mod})
\end{array}
\qquad
\begin{array}{c}
\mathbb{Q} \\
\big\downarrow \\
R
\end{array}
$$

K-Motives: Further Directions

$\rightsquigarrow$

INTERSECTION K-THEORY

TUDOR PĂDURARIU

ORIENTED COHOMOLOGY SHEAVES ON DOUBLE MOMENT GRAPHS

ROSTISLAV DEVYATOV, MARTINA LANINI, AND KIRILL ZAINOULLINE

Representation Theories from Coxeter Groups

M. J. DYER

$\rightsquigarrow$ Soergel conjecture

$\rightsquigarrow$ . . .

THANKS