Universal

Koszul Duality

Today:

Kac-Moody group $\qquad\qquad\qquad\qquad$ Langlands dual

$$\mathcal{G} \quad \rightsquigarrow \quad \widehat{\mathcal{G}}$$

Theme:

$$\text{something attached to } \mathcal{G} \quad \xleftrightarrow{1:1} \quad \text{something else attached to } \widehat{\mathcal{G}}$$

Today:

Koszul duality:

some sheaves on G/B $\overset{1:1}{\longleftrightarrow}$ some other sheaves on $\hat{G}/\hat{B}$

Prediction

Mixed categories, Ext-duality and Representations
(results and conjectures)
by
Alexander Beilinson and Victor Ginsburg

Proof

KATEGORIE $\mathcal{O}$, PERVERSE GARBEN UND MODULN ÜBER DEN KOINVARIANTEN ZUR WEYLGRUPPE

WOLFGANG SOERGEL

Generalizations

KOSZUL DUALITY PATTERNS IN REPRESENTATION THEORY

ALEXANDER BEILINSON, VICTOR GINZBURG, AND WOLFGANG SOERGEL

ENDOSCOPY FOR HECKE CATEGORIES, CHARACTER SHEAVES AND REPRESENTATIONS

GEORGE LUSZTIG AND ZHIWEI YUN

Kazhdan–Lusztig–Basen, unzerlegbare Bimoduln und die Topologie der Fahnenmannigfaltigkeit einer Kac–Moody–Gruppe

Martin Härterich

TOPOLOGICAL APPROACH TO SOERGEL THEORY

ROMAN BEZRUKAVNIKOV AND SIMON RICHE

ON KOSZUL DUALITY FOR KAC-MOODY GROUPS

ROMAN BEZRUKAVNIKOV AND ZHIWEI YUN

Arnaud Eteve

Faisceaux monodromiques, théorie de Deligne-Lusztig, et cohomologie des champs de chtoucas en profondeur 0.

On the relation between intersection cohomology and representation theory in positive characteristic ☆

Wolfgang Soergel

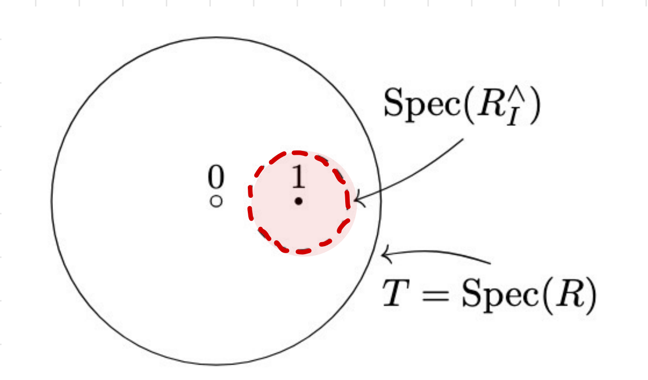
Perverse Monodromic Sheaves

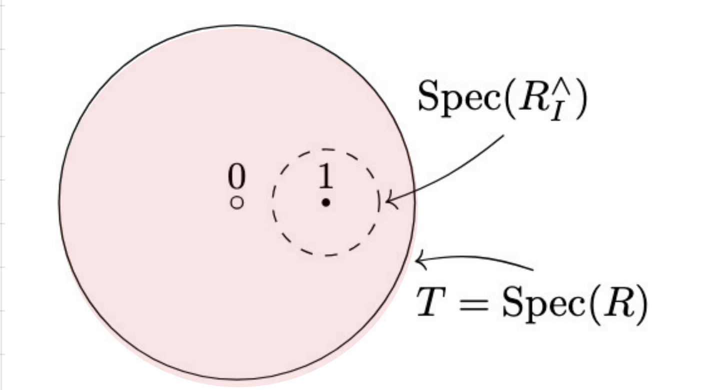
Valentin Gouttard

. . .

Today:

Classical Koszul duality ⇝ Universal Koszul duality





reductive case

Theorem [...]

K-theory Soergel bimodules
Jens Niklas Eberhardt
'22

UNIVERSAL MONODROMIC TILTING SHEAVES
JEREMY TAYLOR
'23

UNIVERSAL KOSZUL DUALITY FOR KAC–MOODY GROUPS
JENS NIKLAS EBERHARDT, ARNAUD ETEVE
'24

$$DK_T(B\backslash \mathcal{G}/B)_{l.c.} \xleftrightarrow{\sim} D_c(\hat{U}\backslash\hat{\mathcal{G}}/\hat{U})_{mon}$$

K-theoretic Hecke category of $\mathcal{G}$

Monodromic Hecke category of $\hat{\mathcal{G}}$

Kac-Moody groups:

$A_{ii} = 2, \quad i \neq j \Rightarrow A_{ij} \leq 0, \quad A_{ij} = 0 \Rightarrow A_{ji} = 0$

character lattice

generalized Cartan matrix

$$\mathcal{D} = \Big(X, \{\overset{\overset{X}{\cup}}{\alpha_i}\}_{i \in I}, \{\overset{\overset{X^\vee}{\cup}}{\alpha_i^\vee}\}_{i \in I}\Big) \qquad A = \big(\langle \alpha_j, \alpha_i^\vee \rangle\big)_{i,j}$$

Kac-Moody datum

roots

coroots

max. torus

Borel

Kac-Moody group

$$T \ltimes U = \underbrace{B \subset \mathcal{G}}_{\text{group ind-schemes}}$$

$X = X(T) = \mathrm{Hom}(T, \mathbb{G}_m)$

faithful if $\{\alpha_i\}$ l.i. $\updownarrow$

$W = N_{\mathcal{G}}(T)/T$

Weyl group

Kac-Moody groups:

Tori:

$$I = \emptyset \Rightarrow \mathcal{G} = \mathcal{B} = \mathcal{T} \cong (\mathbb{G}_m)^{\operatorname{rank} X}$$

Reductive groups:

A non-deg. $\Rightarrow$ $\mathcal{G}$ reductive linear algebraic group

Affine Kac-Moody group:

longest root in $\mathring{\mathcal{D}}$

$$\mathring{\mathcal{D}} = (\mathring{X}, \{\alpha_1, \dots, \alpha_\ell\}, \{\alpha_1^\vee, \dots, \alpha_\ell^\vee\}) \rightsquigarrow \mathcal{D}_{free} = (X = \mathring{X} \oplus \mathbb{Z}\delta, \{\alpha_0 = \delta - \theta, \alpha_1, \dots, \alpha_\ell\}, \{\alpha_0^\vee = -\theta^\vee, \alpha_1, \dots, \alpha_\ell\})$$

$$\mathring{\mathcal{G}} \text{ reductive} \rightsquigarrow \mathcal{G} = \mathring{\mathcal{G}}((t, t^{-1})) \rtimes \mathbb{G}_m$$

loop group

loop rotation

Kac-Moody groups:

Bruhat decomposition:

$$\mathcal{G} = \biguplus_{w \in W} BwB \qquad BwB \cong U_w \times wT \times U \simeq T$$

homotopy equivalent

Langlands dual:

$$\mathcal{D} = (X, \{\alpha_i\}_{i\in I}, \{\alpha_i^\vee\}_{i \in I}) \rightsquigarrow \hat{\mathcal{D}} = (X^\vee, \{\alpha_i^\vee\}_{i\in I}, \{\alpha_i\}_{i\in I})$$

$$\mathcal{G} \rightsquigarrow \hat{\mathcal{G}}$$

$$W = \hat{W}$$

Tori:

torus / $k=\mathbb{C}$ $\qquad\qquad$ dual torus

$$T \qquad\qquad\qquad \hat{T}$$

$$X(T) = \mathrm{Hom}(T, \mathbb{G}_m) = \mathrm{Hom}(\mathbb{G}_m, \hat{T}) = Y(\hat{T})$$

$$\begin{array}{ccc} \| & \qquad\qquad\qquad & \| \\ \mathrm{Irr}(T) = \underset{\text{Line bundles}/\cong}{\mathrm{Pic}}(\cdot/T) & & \underset{\text{topological fundamental group}}{\pi_1}(\hat{T}) \end{array}$$

$$\boxed{K_0(\cdot/T) = R = \mathbb{Z}[\pi_1(\hat{T})]}$$

Equivalence of categories?

$$
\begin{array}{ccccc}
K_0(\cdot/T) & = & R & = & \mathbb{Z}[\pi_1(\hat{T})] \\
\| & & & & \| \\
\mathrm{End}_{??}(??) & & & & \mathrm{End}_{D(\hat{T})}(\mathcal{L})
\end{array}
$$

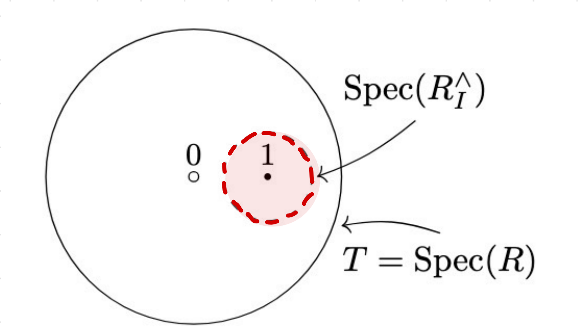
free local system

derived category of sheaves on $\hat{T}(\mathbb{C})^{an}$

$$
??(\cdot/T) = D^b(R) = D_c(\hat{T})_{mon}
$$

$$
D_c(\hat{T})_{mon} = \langle \mathcal{L} \rangle_\Delta \subset D(\hat{T})
$$

Equivalence of categories?



Old solution:

$$R \longrightarrow R_I^\wedge = \prod_i S^i \supset \bigoplus S^i = S = \mathrm{Sym}(X(T))$$

$$S = H^\bullet(\cdot/T)$$

I: Augmentation ideal

$H^\bullet(\cdot/T)$ ← equivariant cohomology ring

$$D^b(\cdot/T) = \mathrm{dgD}(S^\bullet, d=0) \qquad D^b(R_I^\wedge) = D^b(\hat{T})^\wedge_{u\text{-mon}}$$

$\wedge$: completion

u-mon: unipotent monodromy

$$D^b_{mix}(\cdot/T) = D^b(S\text{-mod}^{\mathbb{Z}}) = D^b_{mix}(\hat{T})^\wedge_{u\text{-mon}}$$

D^b_{mix}: "mixed sheaves"

Reduced K-Motives:

INTEGRAL MOTIVIC SHEAVES AND GEOMETRIC REPRESENTATION THEORY

JENS NIKLAS EBERHARDT AND JAKOB SCHOLBACH

The six operations in equivariant motivic homotopy theory

Marc Hoyois [1]

THE CHOW t-STRUCTURE ON THE ∞-CATEGORY OF MOTIVIC SPECTRA

TOM BACHMANN, HANA JIA KONG, GUOZHEN WANG, AND ZHOULI XU

CDH DESCENT IN EQUIVARIANT HOMOTOPY K-THEORY

MARC HOYOIS

$$\mathcal{X} \longmapsto DK_r(\mathcal{X}) = \mathrm{Mod}_{f^*KGL_{S,c=0}}(SH(\mathcal{X}))$$

"nice enough" stack

reduced K-motives

truncated spectrum representing alg. K-theory

stable motivic homotopy category

Main features:

- $f^*, f_*, f_!, f^!$ for representable maps, $\otimes$, Hom
- $\mathbb{A}^1$-homotopy invariant, localisation sequences
- Bott periodicity $(1)[2] = \mathrm{id}$, $f^* = f^!$ for f smooth
- $\mathcal{X}$ smooth, cellular:

$$\boxed{\mathrm{Map}_{DK_r(\mathcal{X})}(1\!\!1, 1\!\!1) = K_0(\mathcal{X})}$$

Equivalence of categories:

$$
\begin{array}{ccccc}
K_0(\cdot/T) & = & R & = & \mathbb{Z}[\pi_1(\hat{T})] \\
\| & & & & \| \\
\mathrm{End}_{DK_r(\cdot/T)}(\mathbb{1}) & & & & \mathrm{End}_{D(\hat{T})}(\mathcal{L})
\end{array}
$$

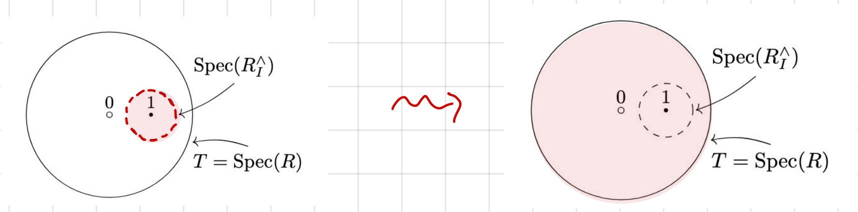
free local system

reduced K-motives on $\cdot/T$

derived category of sheaves on $\hat{T}(\mathbb{C})^{an}$

$$
\boxed{DK_r(\cdot/T)_{l.c.} = D^b(R) = D_c(\hat{T})_{mon}}
$$

$$
\begin{array}{ccc}
\| & & \| \\
\langle \mathbb{1} \rangle_\Delta \subset DK_r(\cdot/T) & & \langle \mathcal{L} \rangle_\Delta \subset D(\hat{T})
\end{array}
$$



K-theoretic Hecke category:

$$B\backslash G/B = \bigsqcup_{w\in W} {}_{B}\backslash BwB/_{B}$$

$$BwB/B \cong \bullet/T$$

Hecke stack

$$\mathcal{H}_{\mathfrak{g}}^{K} = DK_{\Gamma}(B\backslash \mathfrak{g}/B)_{\ell.c.} = \langle \Delta_w \rangle_{\Delta}$$

K-theoretic Hecke category

!-extension of constant object on $B\backslash BwB/B$

K-theoretic Hecke category:

$$\begin{tikzcd}
 & B\backslash G \overset{B}{\times} G/B \arrow[dl, "p_1"'] \arrow[dr, "p_2"] \arrow[rr, "m"] & & B\backslash G/B \\
B\backslash G/B & & B\backslash G/B &
\end{tikzcd}$$

p_1, p_2 not smooth
all maps need to be representable

$$\left.\begin{aligned} A * B &= m_!\,(p_1^* A \otimes p_2^* B) \\ A \overset{!}{*} B &= m_*\,(p_1^! A \otimes p_2^! B) \end{aligned}\right\} \to \text{Monoidal structures}$$

Thm: $* = \overset{!}{*}$ on $\mathcal{H}^K_G$ and $(\mathcal{H}^K_G, *)$ is rigid.

Proof: **A duality formalism in the spirit of Grothendieck and Verdier**

Mitya Boyarchenko[1] and Vladimir Drinfeld[2]

K-theoretic Hecke category:

Thm: Assume that $\{\alpha_i\}$ are l.i. & $\langle -, \alpha_i^\vee \rangle : X \twoheadrightarrow \mathbb{Z}$.

$$
\begin{tikzcd}
\mathbb{K} : \left(\mathcal{H}^K_{\mathfrak{g},*}\right) \arrow[r, "\not\simeq"] & \left(D(R\otimes R), \otimes_R\right) \\
\mathcal{H}^K_{\mathfrak{g},\,\mathrm{pure}} \arrow[u, phantom, "\cup" description] \arrow[r, "\simeq"] \arrow[d, equal] & \mathrm{SBim}^K_{\mathfrak{g}} \arrow[u, phantom, "\cup" description] \arrow[d, equal] \\
\langle E_s \rangle_{*,\oplus} \arrow[d, equal] & \langle R \otimes_{R^s} R \rangle_{\otimes_R,\, \oplus} \\
\mathbb{1}_{B\backslash P_s/B} &
\end{tikzcd}
$$

K-theoretic Hecke category:

Thm: Assume that $\{\alpha_i\}$ are l.i. &, $\langle -, \alpha_i^\vee \rangle : X \twoheadrightarrow \mathbb{Z}$.

$$\mathcal{H}_g^K \xrightarrow{\sim} Ch^b(\mathcal{H}^K_{g,\mathrm{pure}}) \xrightarrow{\sim} Ch^b(SBim_g^K)$$

Bondarko's weight complex functor

Constructible monodromic sheaves:

$$0 \to \pi_1(T) \longrightarrow \mathfrak{t} \xrightarrow{\exp} \hat{T} \longrightarrow 1$$

<u>Thm + Def</u> Let $\hat{T} \curvearrowright X$ (locally compact, Hausdorff)

$$\begin{tikzcd}
 & D(X,\mathbb{Z})_{mon} \subset D(X,\mathbb{Z}) & \\
D(X,\mathbb{Z})_{l.c.} \arrow[ur, "\sim"] & D(X/\mathfrak{t},\mathbb{Z}) \arrow[u, "\wr"] & D(X/(\hat{T},\mathcal{L}_{\hat{T}}),R) \arrow[ul, "\wr"']
\end{tikzcd}$$

$D(X,\mathbb{Z})_{l.c.}$: locally constant on $\hat{T}$-orbits

$D(X/\mathfrak{t},\mathbb{Z})$: $\mathfrak{t}$-equivariant

$D(X/(\hat{T},\mathcal{L}_{\hat{T}}),R)$: $\mathcal{L}_{\hat{T}}$-twisted $\hat{T}$-equivariant

Constructible monodromic sheaves:

weakly constructible + $*$-stalks in $D_{perf}(R)$

$$D_c(X,\mathbb{Z})_{mon} \subset D(X,\mathbb{Z})_{mon}$$

constructible monodromic

<u>Thm</u>: $D_c(-,\mathbb{Z})_{mon}$ has $f^*, f_*, f_!, f^!$.

Monodromic Hecke category:

$$\hat{T} \times \hat{T} \curvearrowright \underset{\uparrow}{\hat{U} \backslash \hat{g} / \hat{U}} = \bigsqcup_{w \in W} \hat{U} \backslash \hat{B} w \hat{B} / \hat{U}$$

$$\hat{U} \backslash \hat{B} w \hat{B} / \hat{U} \cong \hat{T}$$

extended Hecke stack

$$\mathcal{H}_{\hat{g}}^{mon} = D_c(\hat{U} \backslash \hat{g} / \hat{U})_{\hat{T} \times \hat{T}\text{-mon.}}$$

Monodromic Hecke category

$\rightsquigarrow$ rigid monoidal

Monodromic Hecke category:

Thm: Assume that $\{\alpha_i\}$ are l.i. &, $\langle -, \alpha_i^\vee \rangle : X \twoheadrightarrow \mathbb{Z}$.

$$
\begin{array}{ccc}
\mathbb{V} : \left(\mathcal{H}^{\mathrm{mon}}_{\hat{g},*}\right) & \xrightarrow{\not\simeq} & \left(\mathbb{D}(R \otimes R), \underset{R}{\otimes}\right) \\
\cup & & \cup \\
\mathcal{H}^{\mathrm{mon}}_{\hat{g},\mathrm{tilt}} & \xrightarrow{\simeq} & \mathrm{SBim}^{K}_{g} \\
\| & & \| \\
\langle T_s \rangle_{*,\oplus} & & \langle R \underset{R^s}{\otimes} R \rangle_{\underset{R}{\otimes},\oplus}
\end{array}
$$

free monodromic tilting sheaf on $\hat{u} \backslash \hat{P}_s / \hat{u}$

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Perverse Monodromic Sheaves

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Faisceaux monodromiques, théorie de Deligne-Lusztig, et cohomologie des champs de chtoucas en profondeur 0.

Universal Koszul duality:

Thm: Assume that $\{\alpha_i\}$ are l.i. & $\langle -, \alpha_i^\vee \rangle : X \twoheadrightarrow \mathbb{Z}$.

$$\mathcal{H}_{\mathfrak{g}}^{\kappa} \xrightarrow{\simeq} \mathrm{Ch}^b(\mathrm{SBim}_{\mathfrak{g}}^{\kappa}) \xleftarrow{\simeq} \mathcal{H}_{\hat{\mathfrak{g}}}^{\mathrm{mon}}$$

Future:

<u>Conj</u>: "Quantum K-theoretic Satake"

$$DK_{r}(\mathcal{G}\times\mathbb{G}_m\backslash Gr_{\mathcal{G}}) \overset{\sim}{\Leftrightarrow} D_{U_q(\hat{\mathfrak{g}})}(\mathcal{O}_q(\hat{\mathcal{G}}))$$

Type A $\Leftarrow$ **QUANTUM K-THEORETIC GEOMETRIC SATAKE: SL_n CASE** SABIN CAUTIS AND JOEL KAMNITZER + **UNIVERSAL KOSZUL DUALITY FOR KAC–MOODY GROUPS** JENS NIKLAS EBERHARDT, ARNAUD ETEVE

<u>Idea</u> $DM(\mathcal{X},\mathbb{Q}) \rightsquigarrow DK(\mathcal{X}) \rightsquigarrow \cdots \rightsquigarrow DK(n)(\mathcal{X}) \rightsquigarrow \cdots \rightsquigarrow DM(\mathcal{X},\mathbb{Z}/p)$

$$D\Omega(\mathcal{X})$$

Chromatic tower...

$\vdots$

Vielen Dank 💖