

# SEMINAR ON REAL REDUCTIVE GROUPS AND D-MODULES

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## 1. TALKS

1.1.  $\mathrm{SL}_2(\mathbb{C})$ : **Forms, Lie Algebras and Weyl's Unitary Trick.** (Benedek Dombos, 27.10.20) This talk should make us comfortable with our main example  $\mathrm{SL}_2$ .

- Discuss all the different incarnations of  $\mathrm{SL}_2(R)$ : on can see it as an algebraic group, a topological group, a Lie group or an abstract group (also depending on the ring  $R$ ). Discuss  $\mathfrak{sl}_2$ .
- Reductive groups and Lie algebras admit interesting substructures and decompositions. Explain some of them on the example of  $\mathrm{SL}_2$ . In particular the Cartan, Bruhat, Iwasawa and Jordan-Chevalley decomposition.
- The second half of your talk should concern representation theory. Distinguish the different meanings of representations: algebraic, analytic, smooth, etc.
- Explain Weyl's unitary trick and how to identify finite dimensional
  - representations of  $\mathfrak{sl}_2(\mathbb{C})$ ,
  - smooth representations of  $\mathrm{SL}_2(\mathbb{R})$ ,
  - algebraic/holomorphic representations of  $\mathrm{SL}_2(\mathbb{C})$  and
  - unitary representations of  $\mathrm{SU}(2)$ .
- Classify the finite dimensional representations of  $\mathfrak{sl}_2(\mathbb{C})$ . Be sure do draw them as diagrams.
- As an application, mention the Lefschetz decomposition of the cohomology of a Kähler manifold.
- If you have time left explain the finite dimensional *smooth* representations of  $\mathrm{SL}_2(\mathbb{C})$ .

References: are abundant. A nice list of decompositions can be found here. For the unitary trick see [Kna01].

1.2.  $\mathrm{SL}_2(\mathbb{R})$ : **Infinitesimal Equivalence, Harish-Chandra modules and Unitary Representations.** (Hannes Kristinn Árnason, 3.11.20) We start our discussion of *infinite-dimensional* representations of real reductive groups in the example of  $\mathrm{SL}_2(\mathbb{R})$ .

- Explain different notions of infinite dimensional representation: An action on a Banach space, Hilbert space, a locally convex topological vector space, etc.
- Explain the following dilemma. By Gelfand's philosophy, one should study a space  $X$  with the action of a group  $G$  by studying the functions on  $X$ . However, different choices of function space give non-equivalent representations.

- Define the notion of infinitesimal equivalence and Harish-Chandra modules, and explain how this can circumvent the dilemma. Use the example of  $S^1 = \mathrm{SO}(2)$  and Fourier series/polynomials.
- Classify the irreducible admissible  $(\mathfrak{sl}_2(\mathbb{R}), \mathrm{SO}(2))$ -modules and discuss which of them are unitary. Use diagrams instead of formulas whenever possible.

Reference: [Cas] is a really nice essay. If you do the classification of irreducible  $(\mathfrak{g}, K)$ -modules from Section 3, please draw the diagrams from Section 8. For the classification of the unitary representations see Chapter III in [HT92]. Another source is [Ker14].

**1.3.  $\mathrm{SL}_2(\mathbb{R})$ : Principal Series, Discrete Series and Modular Forms.** (Xiaoxiang Zhou, 10.11.20) We find function space interpretations for the Harish-Chandra modules classified above.

- Define the principal series as the space  $\mathrm{Ind}^\infty(\chi)$  for a character  $\chi : B \rightarrow \mathbb{C}^*$ , for  $B \subset \mathrm{SL}_2(\mathbb{R})$  the group of upper triangular matrices.
- Explain how  $\mathrm{Ind}^\infty(\chi)$  is dual to  $\mathrm{Ind}^\infty(\chi^{-1})$ .
- Explain how  $\mathfrak{sl}_2(\mathbb{R})$ ,  $\mathrm{SO}(2)$  and the Casimir element acts.
- Discuss when they are reducible or irreducible, and how all the Harish-Chandra modules from the last talk appear as subquotients.
- Discuss which of them are unitary.
- Draw a picture visualizing the parameters in  $\mathbb{C}$  (purely imaginary line and real unit interval).
- Discuss the discrete series representations. Try to answer the following questions:
  - How can we realize the discrete series in terms of functions on the group or sections in line bundles on the upper half plane  $\mathcal{H}$ .
  - What is the holomorphic and antiholomorphic discrete series. Make sense of the translation *holomorphic*  $\rightleftharpoons$  *highest weight vector*.
- Discuss the relation to modular forms:
  - Recall the definition of a modular form.
  - Explain the relation to functions on the double quotient
 
$$\mathrm{SL}_2(\mathbb{Z}) \backslash \mathrm{SL}_2(\mathbb{R}) / \mathrm{SO}(2).$$
  - It would be desirable to understand a sentence of the form "A modular form (cusp form of weight  $\geq 2$ ) is a highest weight vector of a discrete series summand of  $L^2(\mathrm{SL}_2(\mathbb{Z}) \backslash \mathrm{SL}_2(\mathbb{R}))$ ".

This talk is closely related to the preceding one. Make sure to coordinate your talks. Reference: Again Casselmann's essay [Cas]. In particular: Section 8, 19,20,21. [Kob93] for modular forms and elliptic curves. For the connection to representation theory [Bum97], and for a nice informal account, just concerned with modular forms, see here [Ven].

**1.4. Real Reductive Groups: Definitions, Involutions and Decompositions.** (Berber Lorke, 17.11.20) The goal is to extend the discussion in Talk 1 to arbitrary real reductive groups.

- Recall the definition of a semisimple/reductive complex linear algebraic group.
- Define real semisimple/reductive Lie algebras/groups. Note that there are various definitions of real reductive groups. Discuss the classical groups.

- Explain the Cartan decomposition, compact and split forms.
- Quickly discuss Satake diagrams.
- Explain the Bruhat, Iwasawa and Jordan-Chevalley decompositions. Use the example  $GL_n$  and explain the connection to theorems in linear algebra.

If you are overwhelmed by the content either use a simple definition of real reductive group or restrict to semisimple groups. Reference: You can use [Kna01] which provides many examples. Another source is [Wal88]. Also take a look at Chapter 24 in [Bor91].

**1.5. Real Reductive Groups: Harish-Chandra Modules and Langlands Classification.** (Rostislav Devyatov, 24.11.20) The goal is to extend the construction of all irreducible admissible representations for  $SL_2(\mathbb{R})$  from Talk 2 and Talk 3 to general real reductive groups. The goal is to understand the *Langlands classification* which identifies all irreducible Harish-Chandra modules in terms of quotients of induced unitary representations of parabolic subgroups. There is an analytical and a purely algebraic approach to parabolic induction. The analytic approach needs integrals and functional analysis, while the algebraic approach (due to Zuckermann) relies on derived functors and involves a lot of homological algebra. In this talk we focus on the analytical approach.

- Recall the basic assumptions on our real reductive group. Recall the Langlands decomposition.
- Explain the different approaches to induction. Explain normalized induction.
- Define parabolic induction and principal series modules. Discuss admissibility.
- Determine the central character of an induced representation.
- A theorem of Borel and Harish-Chandra shows that most principal series representations are irreducible. Sketch a proof.
- Sketch a proof that every admissible irreducible occurs as a submodule of an induced representation. Use all “functional analytic” theorems needed as a black-box.
- Sketch the Langlands classification and the idea of a proof.

Use [Ban].

**1.6. Complex Reductive Groups: Harish-Chandra Bimodules and Category  $\mathcal{O}$ .** (Liao Wang, 1.12.20) A complex reductive group  $G$  can be interpreted as a real Lie group by forgetting its complex structure. The representation theory of  $G$  is closely related to category  $\mathcal{O}$  of representations of its Lie algebra  $\mathfrak{g}$  introduced by Bernstein–Gelfand–Gelfand. The goal of this talk is to explain this connection.

- Explain how  $\mathfrak{g}_{\mathbb{C}} \cong \mathfrak{g} \oplus \mathfrak{g}$ . Explain the Cartan decomposition of  $\mathfrak{g}_{\mathbb{C}}$ .
- Show that admissible  $(\mathfrak{g}_{\mathbb{C}}, K)$ -modules can be identified with certain  $U(\mathfrak{g})$ -bimodules called *Harish-Chandra* bimodules.
- Discuss various versions of *Harish-Chandra* bimodules, depending on the action of the left/right action of the center.
- Define category  $\mathcal{O}$ .
- Explain how to pass from Harish-Chandra bimodules to category  $\mathcal{O}$ .
- Explain what this means for character formulas of irreducible admissible  $(\mathfrak{g}_{\mathbb{C}}, K)$ -modules.

See [BGG71a, BGG75, BGG71b, BG80] for the original sources on category  $\mathcal{O}$ . See also the motivation in [Soe92].

**1.7. D-Modules: Linear PDE's, Weyl Algebras and Symbols.** (Thibaud van den Hove, 8.12.20)  $\mathcal{D}$ -modules provide an algebraic (analytic) framework to study linear PDE's over complex varieties (complex manifolds) and generalize vector bundles with flat connection. This talk provides basic definitions and examples of  $\mathcal{D}$ -modules on affine spaces  $\mathbb{A}^n$  and  $\mathbb{G}_m$  and explains the connection to linear PDE's and vector bundles with connection.

- Introduce the Weyl algebra  $\mathcal{D}_{\mathbb{A}^n} = \mathbb{C}[x_i, \partial_i]$ .
- Explain how to transform linear PDE's into modules over  $\mathcal{D}_{\mathbb{A}^n}$  and vice versa.
- Provide some examples of  $\mathcal{D}$ -modules on  $\mathbb{A}^1$  and  $\mathbb{A}^n$ . In particular, the standard, exponential and Dirac- $\delta$  module should be explained.
- Introduce the Fourier transform. Optionally, explain that there is an action of the symplectic group on  $\mathcal{D}_{\mathbb{A}^n}$ . Show some examples.
- Introduce filtrations on  $\mathcal{D}_{\mathbb{A}^n}$  and the notion of a good filtration of  $\mathcal{D}_{\mathbb{A}^n}$ -module. Calculate the characteristic variety of a few examples.
- Provide a differential-geometric intuition for vector bundles with a connection and the notion of flatness. In particular, explain how linear ODE's appear in parallel transport.
- Explain how algebraic vector bundles with connection define a  $D$ -module.
- Discuss the example of the logarithm on  $\mathbb{G}_m$ .

See here. Also [HTT08] as a comprehensive reference. Please discuss with the speakers of the next two talks.

**1.8. D-Modules: Smooth Varieties, Coordinates and Defintions.** (Jiexiang Huang, 15.12.20) In this talk, we extend the definition of  $\mathcal{D}$ -modules to smooth varieties.

- Quickly recall the necessary background for smooth complex varieties, their (co-)tangent space and explain that they admit local coordinates.
- Discuss vector fields and derivations on smooth varieties.
- Define vector bundles with flat connections.
- Provide the definition of  $\mathcal{D}_X$  via vector fields and Grothendieck's construction.
- Explain different perspectives on  $\mathcal{D}_X$ -modules and come back to the example of vector bundles with flat connections.

See here and [HTT08]. Please discuss with the speaker of the previous and next talk.

**1.9. D-Modules: Functorialities and Kashiwara's Theorem.** (Timm Peerenboom, 22.12.20) A particularly powerful feature of  $\mathcal{D}$ -modules are their functorialities. For example, there is a notion of tensor product, internal  $\mathcal{H}om$ -functor and to a morphism  $f : X \rightarrow Y$  one can associate the (exceptional) direct and inverse image functors. This talk aims to construct some of these.

- Quickly explain the tensor product and  $\mathcal{H}om$ -functor for  $\mathcal{D}$ -modules.
- Explain the general philosophy of direct and inverse image functors for modules over rings.
- Explain how this generalizes to quasi-coherent sheaves on varieties.

- Explain, why this does not immediately generalize to  $\mathcal{D}$ -modules.
- Introduce  $\mathcal{D}_{X \rightarrow Y}$  and the direct and inverse image functors for (left, right)  $\mathcal{D}$ -modules.
- Just spend little time on the explanation how one can pass from left to right  $\mathcal{D}$ -modules.
- Explain how the functors behave in examples in local coordinates. Most importantly, consider the case of smooth maps and closed embeddings.
- State Kashiwara's theorem and sketch a proof.
- If there is time, explain that one can derive the direct and inverse image functors. Sketch the relation to de Rham cohomology.

Again, use this and [HTT08]. Please discuss with the speaker of the previous two talks. It is possible to shift items from this talk to the previous one.

1.10.  $\mathbb{P}^1$ : **Line bundles, Representations and Springer Resolution.** (Daniel Bermudez, 12.1.21) This is the first in a series of talk discussing the localization theorem in the case of  $\mathfrak{sl}_2$ . The goal is to introduce the necessary background on the geometry of  $\mathbb{P}^1$ , and first applications to representation theory.

- Introduce  $\mathbb{P}^1$  as quotient and by charts. Discuss how to obtain line bundles via balanced products and explicitly in charts.
- Compute the global sections and state the Borel–Weil theorem. Quickly explain Serre duality and the connection to the Borel–Weil–Bott theorem.
- Define the cotangent bundle  $T^*\mathbb{P}^1$  of  $\mathbb{P}^1$ , provide different realisations.
- Explain how to define  $\mathcal{N}$  in terms of the characteristic polynomial.
- Discuss the isomorphism

$$S(\mathfrak{sl}_2(\mathbb{C}))/S(\mathfrak{sl}_2(\mathbb{C}))_+^{\mathrm{SL}_2(\mathbb{C})} \rightarrow \mathbb{C}[\mathcal{N}].$$

- Define the Springer resolution and discuss its normality.

See [CG10], [HTT08] for general reference and here and [Rom20], for very concrete calculations. Please discuss with the speakers of the next two talks.

1.11.  $\mathbb{P}^1$ : **Global Differential Operators, Global Sections and the Localisation Theorem.** (Jon Miles, 19.1.21) The goal is to prove the localisation theorem for  $\mathbb{P}^1$ .

- Explain the PBW theorem and the Harish-Chandra isomorphism in the example of  $\mathfrak{sl}_2(\mathbb{C})$ .
- Recall the filtration  $\mathcal{D}_X$ , its associated graded and the connection to the cotangent bundle.
- Explain the map

$$U(\mathfrak{sl}_2(\mathbb{C})) \rightarrow D_{\mathbb{P}^1} = \Gamma(\mathbb{P}^1, \mathcal{D}_{\mathbb{P}^1}).$$

- Prove that it induces an isomorphism

$$U(\mathfrak{sl}_2(\mathbb{C}))/Z_+ \rightarrow D_{\mathbb{P}^1}$$

using the results on the associated graded from the last talk.

- Sketch a proof that the global section functor induces an equivalence between (certain)  $\mathcal{D}_{\mathbb{P}^1}$ - and  $D_{\mathbb{P}^1}$ -modules (localisation theorem). There should be a way to do this very explicitly for  $\mathbb{P}^1$  without using the whole general machinery.

For references, see Talk 10. Please discuss with the speaker of the previous and next talk.

1.12.  $\mathbb{P}^1$ : **Principal Series and Discrete Series via D-Modules.** (Konrad Zou, 26.1.21) The goal of this talk is to construct Harish-Chandra modules for  $\mathrm{SL}_2(\mathbb{R})$  using D-modules.

- Define twisted D-modules on  $\mathbb{P}_1$ .
- Explain (without proofs) how the results from the last to talks generalize to twisted D-modules.
- Discuss the action of  $K_{\mathbb{C}} = \mathbb{G}_m$  on  $\mathbb{P}_1$ .
- Define  $\mathbb{G}_m$ -equivariant D-modules.
- Define Verma, principal series and discrete series D-modules using the push-forward functors discussed in talk 9. Make sure to do this as explicit as possible.
- Compare the global to the principal and discrete series Harish-Chandra modules discussed in talk 2.

Use [Rom20]. Also, please discuss with the previous two speakers.

1.13. **Flag varieties: Localisation Theorem.** (Mingyu Ni, 2.2.21) The goal of this talk is explain the localisation theorem for complex reductive Lie algebras. Recall the results from the last three talks and elaborate how to extend them. Use this here.

1.14. **Flag varieties: K-orbits, Local Systems and Langlands Classification.** (9.2.21) The goal of this last talk is to obtain another Langlands Classification for irreducible Harish-Chandra modules for real reductive groups, this time using  $D$ -modules. The topic is quite difficult and this talk is more expository than proof-based.

- Recall how the localisation theorem for equivariant and twisted  $D$ -modules allows to study  $(\mathfrak{g}, K)$ -modules geometrically.
- Explain how to understand admissibility geometrically.
- Sketch the Riemann–Hilbert correspondence and the classification of simple perverse sheaves.
- Sketch the Langlands Classification.
- Come back to the  $\mathrm{SL}_2(\mathbb{R})$  and  $\mathrm{SL}_2(\mathbb{C})$  examples.

See [HTT08].

#### REFERENCES

- [Ban] E.P. van den Ban, *Induced representations and the langlands classification*, Lecture Notes <http://dspace.library.uu.nl/handle/1874/1625>.
- [BG80] J. N. Bernstein and S. I. Gelfand, *Tensor products of finite- and infinite-dimensional representations of semisimple Lie algebras*, *Compos. Math.* **41** (1980), 245–285 (English).
- [BGG71a] I. N. Bernstein, I. M. Gel’fand, and S. I. Gel’fand, *Structure of representations generated by vectors of highest weight*, *Funct. Anal. Appl.* **5** (1971), 1–8 (English).
- [BGG71b] ———, *Structure of representations generated by vectors of highest weight*, *Funct. Anal. Appl.* **5** (1971), 1–8 (English).
- [BGG75] I. N. Bernstein, I. M. Gelfand, and S. I. Gelfand, *Differential operators on the base affine space and a study of  $\mathfrak{g}$ -modules*, *Lie Groups Represent., Proc. Summer Sch. Bolyai Janos Math. Soc., Budapest 1971, 21-64 (1975).*, 1975.
- [Bor91] Armand Borel, *Linear algebraic groups*, Springer New York, 1991.

- [Bum97] Daniel Bump, *Automorphic forms and representations*, Cambridge Studies in Advanced Mathematics, vol. 55, Cambridge University Press, Cambridge, 1997. MR 1431508
- [Cas] Bill Casselmann, *Admissible representations of  $SL(2, \mathbb{R})$* , Essay on <https://www.math.ubc.ca/~cass/research/essays.html>.
- [CG10] Neil Chriss and Victor Ginzburg, *Representation theory and complex geometry*, Birkhäuser Boston, 2010.
- [HT92] Roger Howe and Eng-Chye Tan, *Nonabelian harmonic analysis*, Universitext, Springer-Verlag, New York, 1992, Applications of  $SL(2, \mathbf{R})$ . MR 1151617
- [HTT08] Ryoshi Hotta, Kiyoshi Takeuchi, and Toshiyuki Tanisaki (eds.), *D-modules, perverse sheaves, and representation theory*, Birkhäuser Boston, 2008.
- [Ker14] Matt Kerr, *Notes on the representation theory of  $sl_2(r)$* , Hodge Theory, Complex Geometry, and Representation Theory **608** (2014), 173.
- [Kna01] Anthony W. Knaapp, *Representation theory of semisimple groups*, Princeton Landmarks in Mathematics, Princeton University Press, Princeton, NJ, 2001, An overview based on examples, Reprint of the 1986 original. MR 1880691
- [Kob93] Neal Koblitz, *Introduction to elliptic curves and modular forms*, second ed., Graduate Texts in Mathematics, vol. 97, Springer-Verlag, New York, 1993. MR 1216136
- [Rom20] Anna Romanov, *Four examples of beilinson-bernstein localization*, 2020.
- [Soe92] Wolfgang Soergel, *The combinatorics of Harish-Chandra bimodules*, J. Reine Angew. Math. **429** (1992), 49–74 (English).
- [Ven] T.N. Venkataramana, *Classical modular forms*, Lecture Notes, see [http://users.ictp.it/~pub\\_off/lectures/lns021/Venkataramana/Venkataramana.pdf](http://users.ictp.it/~pub_off/lectures/lns021/Venkataramana/Venkataramana.pdf).
- [Wal88] Nolan R Wallach, *Real reductive groups i*, Academic press, 1988.