HW9

1| Let V be an inner product space over $F = \mathbb{R}$. Recall the map

$$\ell : V \longrightarrow V^*, \quad x \longmapsto (V \to F,\ y \mapsto \langle x, y \rangle).$$

So $\ell(x)(y) = \langle x, y \rangle \ \forall x, y \in V$. Let $\beta = \{v_1, \dots, v_n\}$ be a basis of V. Show that

$$\boxed{v_i^* = \ell(v_i) \ \forall 1 \le i \le n \iff \beta \text{ is orthonormal}}$$

where $\beta^* = \{v_1^*, \dots, v_n^*\}$ is the dual basis.

2| Let V be a finite dimensional inner product space. Show

(1) $(W^\perp)^\perp = W$ for all subspaces $W \subset V$

(2) $W^\perp \supset U^\perp$ for all $U \subset W \subset V$

(3) $(W \cap U)^\perp = W^\perp + U^\perp$ for all subspaces $U, W \subset V$

(4) $W^\perp \cap U^\perp = (V + U)^\perp$ for all subspaces $U, W \subset V$

3| Let $V = F^{2\times 2}$ equipped with the Frobenius inner product

$$\langle A, B \rangle = \operatorname{tr}(B^\dagger A).$$

Let $W = \operatorname{span}\left(\left\{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right\}\right)$. Determine $W^{\perp}$.
For a general matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, determine $\operatorname{pr}_W(A)$, $\operatorname{pr}_{W^{\perp}}(A)$.

4| Let $V = F^3$ be a vectorspace over a field F.
Let $W = \operatorname{span}\left(\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right\}\right) = xy$-plane. Determine the set of all complements of W.

In general, can you parametrize the set of all complements of $F^m \subset F^n$, $m \leq n$?