HW5

1| Let F be a field, n a positive integer. Denote by

$$F[x]_{\leq n} = \left\{ \sum_{i=0}^{n} a_i x^i \;\middle|\; a_i \in F \right\}$$

the space of polynomials of degree $\leq n$ over F. Let $\beta = \{1, x, \dots, x^n\}$ be the std. basis of $F[x]_{\leq n}$ and

$$D : F[x]_{\leq n} \to F[x]_{\leq n}, \quad f \mapsto \frac{d}{dx} f.$$

Compute $[D]^{\beta}_{\beta}$.

Now assume that $F = \mathbb{F}_2$. Compute $\ker([D]^{\beta}_{\beta})$ and $\ker(D)$.

2| Let $X = \{1,2,3\}$, $Y = \{1,2,3,4\}$. $f : X \to Y$, $1 \mapsto 1$, $2 \mapsto 1$, $3 \mapsto 3$.

Let F be a field and $\beta = \{\mathbb{1}_1, \mathbb{1}_2, \mathbb{1}_3\} \subset \mathrm{Fun}(X,F)$, $\gamma = \{\mathbb{1}_1, \mathbb{1}_2, \mathbb{1}_3, \mathbb{1}_4\} \subset \mathrm{Fun}(Y,F)$ be the std. bases. Compute: $[f^*]^{\beta}_{\gamma}$, $[f_!]^{\gamma}_{\beta}$, $\ker(f^*)$, $\ker(f_!)$, $\mathrm{im}(f^*)$, $\mathrm{im}(f_!)$, where

$$f^* : \mathrm{Fun}(Y,F) \to \mathrm{Fun}(X,F), \quad f^*(\alpha) = \alpha f$$

$$f_! : \mathrm{Fun}(X,F) \to \mathrm{Fun}(Y,F), \quad (f_!(\alpha))(y) = \sum_{x \in f^{-1}(\{x\})} \alpha(x).$$

3| Let V be a f.d. v.s. over some field F, $\beta \subset V$ some basis. Compute $[\mathrm{id}_V]^{\beta}_{\beta}$.

4) Let V be a vector space over a field F and $\beta \subset V$ be a basis. Show that

$$V^* \longrightarrow \mathrm{Fun}(\beta, F), \quad \lambda \longmapsto \lambda|_\beta$$

is an isomorphism, where $\lambda|_\beta$ denotes the restriction of λ to β for $\lambda \in V^*$.