HW3

1.| Let V/F be a vector space, $L, \mathcal{G} \subset V$ finite subsets such that
(1) $\text{span}(\mathcal{G}) = L$ (2) L linearly independent.
Present an algorithm (in words or pseudo code) which constructs a subset $H \subset \mathcal{G}$, such that $|\mathcal{G} - H| = |L|$ and $\text{span}(H \cup L) = V$.
Hint: Use the proof of the replacement theorem).

2.| Let V/F be a finite dimensional vector space. Let $W \subset V$ be a subspace.

(a) Show that W is finite dimensional
(b) Show that there are $\beta_1, \beta_2 \subset V$ such that
  (a) β_1 is a basis of W
  (b) $\beta_1 \cap \beta_2 = \emptyset$ and $\beta = \beta_1 \cup \beta_2$ is a basis of V
  (c) In the notation of (b), let $\beta_2 = \{u_1, \dots, u_k\}$, and denote by $[u_i] = u_i + W \in V/W$, their cosets in V/W. Show that $\{[u_1], \dots, [u_k]\} \subset V/W$ is linearly independent.

3.| Let $T: V \to W$ be a homomorphism of vector spaces over F. Show that

(a) $\text{im}(T) \subset W$ and $\ker(T) \subset V$ are subspaces
(b) Let $S: W \to U$ be another homomorphism. Show that $ST: V \to U$ is also a homomorphism.
(c) Assume that T is surjective, and $V = \text{span}(\{u_1, \dots, u_n\})$. Show that $W = \text{span}(\{T(u_1), \dots, T(u_n)\})$

4.| Construct 5 unique examples of linear maps.