Kac-Moody Lie Algebras and Flag Varieties

Lecture 1:

Organisation:

- 2 lectures/week
- Oral exam (date TBD)
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Contents:

Prerequisites:

- Semi-simple Lie algebras + f.d rep's.
- basic AG: basics on complex proj. varieties

Sources:

- Infinite Dimensional Lie Algebras, Kac
- Infinite Dimensional Lie Algebras, Wakimoto
- Kac-Moody Groups, their flag var. and rep.th., Kumar

⋮

⇝ see website

0. Semisimple Lie Algebras: Recall

Global Assumption: Everything $/ \mathbb{C}$

0.1. Lie Algebras

<u>Def.</u> A <u>Lie algebra</u> $\mathfrak{g}$ is a vector space $/\mathbb{C}$ with a <u>Lie bracket</u> $[\ ,\ ]: \Lambda^2 \mathfrak{g} \to \mathfrak{g}$, s.t.

$$\mathrm{ad}(x): \mathfrak{g} \to \mathfrak{g}, \quad y \mapsto [x,y]$$

is a <u>derivation</u> for each x, that is,

$$\mathrm{ad}(x)([y,z]) = [\mathrm{ad}(x)(y), z] + [y, \mathrm{ad}(x)(z)] \quad \square$$

<u>Example</u> (1) Let A be an associative algebra. Then $[x,y] = xy - yx$ turns A into a Lie algebra.

(2) For V a v.s. and $A = \mathrm{End}(V)$, we obtain

$$\mathfrak{gl}(V) = \mathrm{End}(V) + [\,,\,]$$

(3) Let G be a complex Lie/algebraic group. Then

$$\mathfrak{g} = \mathrm{Lie}(G) = T_e X$$

is a Lie algebra

(5) The set of derivations $\mathrm{Der}(\mathfrak{g}) \subset \mathfrak{gl}(\mathfrak{g})$ is a Lie algebra.

(6) $\mathfrak{gl}_n, \mathfrak{sl}_n, \mathfrak{so}_n, \mathfrak{sp}_n, \ldots$

(6) The $2n+1$-dim. Lie algebra $\mathfrak{g}$, spanned by

$u_1, \dots, u_n, v_1, \dots, v_n, z$ with bracket defined by

$$[u_i, v_j] = \delta_{ij} z \quad , \quad [\mathfrak{g}, z] = 0$$

$$(\Rightarrow [u_i, u_j] = [v_i, v_j] = 0)$$

is called the <u>Heisenberg algebra</u>

(7) The most important example:

$$sl_2(\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \right\} = \left\langle e = \begin{pmatrix} & 1 \\ & \end{pmatrix}, h = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, f = \begin{pmatrix} & \\ 1 & \end{pmatrix} \right\rangle \subset gl(\mathbb{C}^2)$$

It has relations: $[h, e] = 2e, \ [h, f] = -2f, \ [e, f] = h$

$\square$

Def. B: A representation V of $\mathfrak{g}$ is a map (of Lie algebras), $\mathfrak{g} \to \mathfrak{gl}(V)$ $\square$

Example B (1) $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$, $x \mapsto \mathrm{ad}(x)$ defines the <u>adjoint representation</u>.

We have $\mathrm{ad}(\mathfrak{g}) \subset \mathrm{der}(\mathfrak{g})$.

(2) For $\mathfrak{sl}_2(\mathbb{C})$ we obtain (with the basis e, h, f)

$$\mathrm{ad}(e) = \begin{pmatrix} 0 & -2 & 0 \\ & 0 & 1 \\ & & 0 \end{pmatrix} \quad \mathrm{ad}(h) = \begin{pmatrix} 2 & & \\ & 0 & \\ & & -2 \end{pmatrix} \quad \mathrm{ad}(f) = \begin{pmatrix} 0 & & \\ -1 & 0 & \\ 0 & 2 & 0 \end{pmatrix}$$

(3) Let $\mathfrak{g} = \langle u, v, z \rangle$ be the 3-dim Heisenberg algebra. Then we obtain a faithful representation:

$$u \mapsto \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad v \mapsto \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad z \mapsto \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(Generalize this from 3 to $2n+1$)

(4) For the $2n+1$-dim. Heisenberg algebra, we obtain the <u>Schrödinger representation</u> on $\mathbb{C}[x_1, .., x_n]$ via

$u_i \mapsto \frac{\partial}{\partial x_i}$, $v_i \mapsto x_i$, $z \mapsto id$ $\square$

0.2 Universal enveloping algebra

Let $\mathfrak{g}$ be a Lie algebra.

<u>Def</u>: The <u>universal enveloping algebra</u> $U(\mathfrak{g})$ of $\mathfrak{g}$ is the associative algebra defined by

$$T^\bullet V \big/ (x \otimes y - y \otimes x = [x, y]) \qquad \square$$

<u>Rem</u>: We omit the $\otimes$ in the notation for elements in $T^\bullet V$

<u>Prop</u>: $U(\mathfrak{g})$ is determined by the universal property

$$\begin{tikzcd}
\mathfrak{g} \arrow[r] \arrow[dr] & U(\mathfrak{g}) \arrow[d, dotted, "\exists!"] \\
& A
\end{tikzcd}$$

Lie alg. map

unital associative algebra $\square$

<u>Thm</u>: (Poincaré - Birkhoff - Witt) Denote by $\mathcal{U}_n(\mathfrak{g}) \subset \mathcal{U}(\mathfrak{g})$

the image of $\bigoplus_{i \leq n} T^i(\mathfrak{g})$ under $T^{\bullet}\mathfrak{g} \twoheadrightarrow \mathcal{U}(\mathfrak{g})$. Then

(1) $\mathcal{U}_n(\mathfrak{g}) \subset \mathcal{U}_m(\mathfrak{g})$ for $n \leq m$

(2) $\mathcal{U}_n(\mathfrak{g})\,\mathcal{U}_m(\mathfrak{g}) \subset \mathcal{U}_{n+m}(\mathfrak{g}) \quad \forall n, m$

(3) $\mathrm{gr}\,\mathcal{U}(\mathfrak{g}) = \bigoplus_n \mathcal{U}_n(\mathfrak{g})/\mathcal{U}_{n-1}(\mathfrak{g})$ is

a commutative graded algebra

(4) The natural map $\mathrm{Sym}(\mathfrak{g}) \to \mathrm{gr}\,\mathcal{U}(\mathfrak{g})$ is

an isom. of graded algebras $\square$

The Thm. implies that as vector space, $\mathcal{U}(\mathfrak{g})$ and $\mathrm{Sym}(\mathfrak{g})$

behave the same.

0.3 Automorphisms

Let $\dim \mathfrak{g} < \infty$

<u>Def</u> The group of (inner) automorphism of $\mathfrak{g}$ is

$$\begin{array}{ccccc} \mathrm{Int}(\mathfrak{g}) & \subset & \mathrm{Aut}(\mathfrak{g}) & \subset & GL(\mathfrak{g}) \\ \| & & \uparrow & & \\ \langle e^{\mathrm{ad}(x)} \mid x \in \mathfrak{g} \rangle & & \text{Lie algebra autom.} & & \end{array}$$

$\square$

<u>Prop</u>: $\mathrm{Int}(\mathfrak{g})$, $\mathrm{Aut}(\mathfrak{g})$ are Lie groups with Lie algebras

$$\begin{array}{ccccc} \mathrm{Lie}(\mathrm{Int}(\mathfrak{g})) & \to & \mathrm{Lie}(\mathrm{Aut}(\mathfrak{g})) & \to & \mathfrak{gl}(\mathfrak{g}) \\ \wr\| & & \wr\| & & \\ \mathfrak{g}/\mathfrak{z} & \xrightarrow{\mathrm{ad}} & \mathrm{Der}(\mathfrak{g}) & & \end{array}$$

where $\mathfrak{z} = \ker(\mathrm{ad})$

$\square$

Remark: (1) The exponential $e^X = \sum_{j=0}^{\infty} \frac{1}{j!} X^j$ for $X \in \mathrm{End}(V)$ works well when either

(a) $\dim V < \infty$

(b) X is locally nilpotent, that is $\forall v \in V \; \exists n \geq 0$, s.t. $X^n v = 0$

(2) Under this conditions $e^{\mathrm{ad}(X)} = \mathrm{Ad}(e^X)$ and similar standard formulas hold $\square$

Lecture 2

0.4. Representation theory of $\mathfrak{sl}_2$

We recall the rep.-th. of

$$\mathfrak{sl}_2(\mathbb{C}) = \left\langle e=\begin{pmatrix} & 1\\ & \end{pmatrix}, h=\begin{pmatrix}1 & \\ & -1\end{pmatrix}, f=\begin{pmatrix} & \\ 1 & \end{pmatrix}\right\rangle$$

$$[h,e]=2e, \quad [h,f]=-2f, \quad [e,f]=h$$

<u>Lemma</u>: The following relations hold in $U(\mathfrak{sl}_2(\mathbb{C}))$

(1) $[h,e^k] = 2k\, e^k, \qquad [h,f^k] = -2k\, f^k$

(2) $[e,f^k] = k\, f^{k-1}(h+1-k) = k\,(h+k-1)\, f^{k-1}$

$$[f,e^k] = -k\, e^{k-1}(h+k-1) = -k\,(h+1-k)\, e^{k-1}$$

<u>Proof</u>: (1) $[h,e^k] = [h,e]e^{k-1} + e\,[h,e^{k-1}]$

$$= \underbrace{2e}_{\text{relation}}\, e^{k-1} + e\, \underbrace{2(k-1)\, e^{k-1}}_{\text{induction}} = 2k\, e^k$$

(2) $[e, f^k] = [e,f]f^{k-1} + f \cdot [e, f^{k-1}]$

$$= h f^{k-1} + f(k-1)(h+k-2)f^{k-2}$$

$$\overset{fh = hf+2f}{=} h f^{k-1} + (k-1) h f^{k-1} + (k-1) 2 f^{k-1} + (k-1)(k-2) f^{k-1}$$

$$= k(h+k-1) f^{k-1}$$

The other equations are analogous $\square$

<u>Prop.</u> Let V be a $\mathfrak{sl}_2$-rep. with $v \in V$, $\lambda \in \mathbb{C}$ such that $hv = \lambda v$. Let $v_j = 1/j! \, f^j v$.

Then

$$h v_j = (\lambda - 2j) v_j$$

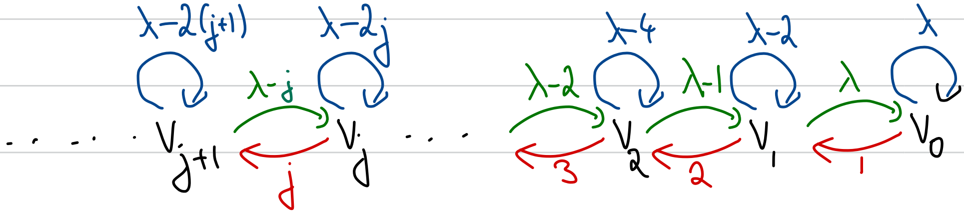
If $ev = 0$, then we have

$$e v_j = (\lambda - j + 1)\, v_{j-1}$$

Proof: Use Lemma (1) and (2) $\square$

Rem: (1) If $hv = \lambda v$ and $ev = 0$, v is called a highest weight vector (of weight λ)

(2) One can visualise this as:



where $e : \longrightarrow$, $h : \circlearrowright$, $f : \longleftarrow$

Def A Let $\lambda \in \mathbb{C}$, $\mathfrak{b} = \langle h, e \rangle \subset sl_2$. Then

$$M(\lambda) = U(sl_2) \underset{U(\mathfrak{b})}{\otimes} \mathbb{C}_\lambda$$

is the Verma module (or universal highest weight module) for sl_2 of highest weight λ. Here $\mathfrak{b}$ acts on $\mathbb{C}_\lambda = \mathbb{C}$ via $ev = 0$, $hv = \lambda v$ $\forall v \in \mathbb{C}_\lambda$ $\square$

Prop. B: Let $\mathfrak{n}^- = \langle f \rangle \subset sl_2$, $\lambda \in \mathbb{C}$ and $v^+ \in \mathbb{C}_\lambda \setminus \{0\}$.

(1) $\mathbb{C}[f] = Sym(\mathfrak{n}^-) = U(\mathfrak{n}^-) \longrightarrow M(\lambda)$, $1 \mapsto v^+$

is an isomorphism of $U(\mathfrak{n}^-)$-modules. In particular $\{ v_j = \frac{1}{j!} f^j v^+ \}_{j \geq 0}$ is a basis of $M(\lambda)$

(2) Let v_0 be a highest weight vector in V of weight λ. Then there is a unique map

$$M(\lambda) \longrightarrow V, \quad v^+ \longmapsto v_0$$

<u>Proof</u> Exercise $\square$

<u>Thm:</u> Let $\lambda \in \mathbb{C}$. Then

(1) $M(\lambda)$ is irreducible $\iff \lambda \notin \mathbb{Z}_{\geq 0}$

(2) Let $\lambda \in \mathbb{Z}_{\geq 0}$, then there is an injective map

$$\begin{array}{ccccc} M(-\lambda-2) & \hookrightarrow & M(\lambda) & \twoheadrightarrow & L(\lambda) \\ w^+ & \longmapsto & \frac{1}{(\lambda+1)!} f^{(\lambda+1)} v^+ & & \end{array}$$

and the quotient is a irr. f.d. representation $L(\lambda)$

of dimension $\lambda+1$. Here w^+, v^+ are the highest weight vectors of $M(-\lambda-2)$ and $M(\lambda)$, respectively.

(3) Let L be a simple, f.d. rep. of $sl_2(\mathbb{C})$. Then there is a (unique up to scalar) highest weight vector $v_0 \in L$. Then v_0 has weight $\lambda \in \mathbb{Z}_{\geq 0}$ and the natural map $M(\lambda) \to L(\lambda)$ yield an iso. $L(\lambda) \xrightarrow{\sim} L$ $\square$

Pf: Denote a basis of $M(\lambda)$ as in Prop. B. Then by Prop A,

(∗) $e v_{j+1} = (\lambda - j) v_j = 0 \iff \lambda = j$

(1) "$\Leftarrow$" Assume that $\lambda \notin \mathbb{Z}_{\geq 0}$ and let $x \in M(\lambda) \setminus \{0\}$.

Let $k \in \mathbb{Z}$, s.t. $x = a_k v_k + \sum_{j<k} a_j v_j$ with

$a_k \neq 0$. Then by (∗) $e^k x = a v^+$ where $a \neq 0$

Since v^+ generates $M(\lambda)$, so does x. Hence

$M(\lambda)$ is irreducible

(2) + (1) "$\Rightarrow$" If $\lambda \in \mathbb{Z}_{\geq 0}$, by (∗) $v_{\lambda+1}$

highest weight vector of (by Prop A) weight

$\lambda - 2(\lambda+1) = -\lambda - 2$. Hence, there is a unique map

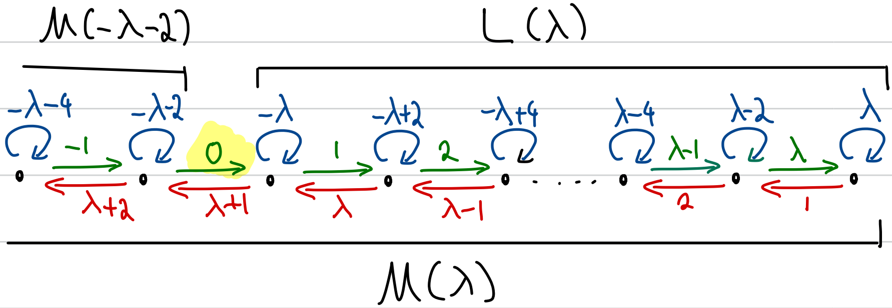
$$M(-\lambda-2) \longrightarrow M(\lambda), \quad w^+ \mapsto v_{\lambda+1}.$$

Since $-\lambda-2 \notin \mathbb{Z}_{\geq 0}$, $M(-\lambda-2)$ is irreducible and the map is hence injective.

By a similar argument as in the proof of (1)"$\Leftarrow$", the quotient $L(\lambda) = M(\lambda)/M(-\lambda-2)$ is irreducible, and $L(\lambda)$ has basis $\bar{v}_0, \dots, \bar{v}_\lambda$.

(3) Exercise. Hint: decompose L in eigenspaces of h. $\square$

<u>Sketch</u> For $\lambda \in \mathbb{Z}_{\geq 0}$, we obtain:



Example: (1) $L(0) = \mathbb{C}_0$ = trivial

$L(1) = \mathbb{C}^2$ = fundamental

$L(2) = \mathfrak{sl}_2$ = adjoint

In general $L(\lambda) = \mathrm{Sym}^\lambda(\mathbb{C}^2) = k[x,y]_\lambda$

homogeneous polynomials of degree λ $\triangleright$

Lecture 3

We will study Lie algebras via copies of $\mathfrak{sl}_2$'s they contain:

Def B: Let $\mathfrak{g}$ be a Lie algebra. A choice of elements $e, h, f \in \mathfrak{g}$ with $[e,f]=h$, $[h,e]=2e$, $[h,f]=-2f$ is called an $\mathfrak{sl}_2$-triple in $\mathfrak{g}$

0.5 Killing form

Let $\mathfrak{g}$ be a Lie algebra

<u>Def</u>: (1) A symmetric form $B: \mathfrak{g} \otimes \mathfrak{g} \to k$ is called <u>$\mathfrak{g}$-invariant</u> if

$$B([x,y],z) = B(x,[y,z]) \quad \forall x,y,z \in \mathfrak{g}$$

(2) If $\rho: \mathfrak{g} \to \mathfrak{gl}(V)$ is a f.d. rep. of $\mathfrak{g}$, we denote

$$(x,y)_V = \mathrm{tr}(\rho(x)\rho(y))$$

If $\rho = \mathrm{ad}$, $(,) = (,)_V$ is the <u>Killing form</u>

(3) Assume that $\mathfrak{g}$ is f.d. and B_V as in (2) is non-deg. Pick a pair of dual bases $\{x_i\}, \{x^i\}$ of $\mathfrak{g}$, that is, $(x_i, x^j)_V = \delta_{ij}$. Then

the <u>Casimir operator</u> is defined by

$$C_V = \sum x_i x^i$$

<u>Lemma</u> (1) $(\, , \,)_V$ is a $\mathfrak{g}$-invariant form.

(2) C_V does not depend on the choice of basis.

(3) $C_V \in Z(\mathfrak{g}) = Z(U(\mathfrak{g}))$

<u>Proof</u>: Omitted $\square$

<u>Example</u> Let $\mathfrak{g} = \mathfrak{sl}_2$, $B(X,Y) = tr(XY)$, then we get dual bases $\{e, h, f\}$, $\{f, \frac{1}{2}h, e\}$ and the Casimir operator

$$C = ef + \frac{1}{2}h^2 + fe = \frac{1}{2}h^2 + h + 2fe$$

By Schur's Lemma C act by a scalar on $L(\lambda)$. Let $v^+ \in L(\lambda)$ the highest weight vector, then $C v^+ = (\frac{1}{2}h^2 + h + fe) v^+ = (\frac{1}{2}\lambda^2 + \lambda) \cdot v^+$

Since $\lambda \mapsto \frac{1}{2}\lambda^2 + \lambda$ is injective for $\lambda \in \mathbb{Z}_{\geq 0}$, we can decompose f.d. sl_2-rep's in isotypic components using eigenspaces of C $\square$

0.6 Semisimple Lie Algebras - Definitions

Def/Thm: A f.d. Lie algebra $\mathfrak{g}$ is called semisimple if (one of) the following equivalent statements is fulfilled:

(1) $\mathfrak{g}$ has no Abelian ideals

(2) $\mathfrak{g}$ has no solvable ideals

(3) The Killing form $(\, , \,)$ is non-deg.

(4) $\mathfrak{g}$ is a sum of simple Lie algebras

Moreover $\mathfrak{g}$ is called reductive if $\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{z}$ with $\mathfrak{g}'$ semisimple and $\mathfrak{z}$ Abelian

Proof: Idea: $\operatorname{rad}((\,,\,)) \subset \operatorname{rad}(\mathfrak{g})$ ....

radical of Killing form — max solvable ideal in $\mathfrak{g}$ $\square$

Thm: Let $\mathfrak{g}$ be a f.d. Lie algebra. Then

$\mathfrak{g}$ semisimple $\iff$ every f.d. rep. is a sum of irreducibles

Proof: Sketch:

"$\Leftarrow$" Decompose $V = \mathfrak{g}$ in irreducibles ...

"$\Rightarrow$" It suffices to show: $W \mapsto \operatorname{Hom}_{\mathfrak{g}}(V, -)$ is exact for V, W f.d. rep's.

To see this, use

$$\mathrm{Hom}_{\mathfrak{g}}(V, W) = \mathrm{Hom}(V, W)^{\mathfrak{g}}.$$

Since Hom is exact, it suffices to show that

$$V \mapsto V^{\mathfrak{g}} = \mathrm{Hom}(\mathrm{triv}, V)$$

is exact, or that $\mathrm{Ext}^1(\mathrm{triv}, V) = 0$, or that every s.e.s. of the form

$$0 \to V \to E \to \mathrm{triv} \to 0$$

splits. If $V \neq \mathrm{triv}$ is irreducible, there is a splitting $E = V \oplus \ker(C_V)$, where $\ker(C_V) \cong \mathrm{triv}$ and C_V is the Casimir of V.

If $V = \text{triv}$, $\rho_E(\mathfrak{g}) \subset \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}$ is solvable. Since $\mathfrak{g}$ is semisimple, $\rho_E(\mathfrak{g}) = 0$, so $E = \text{triv}^{\oplus 2}$.

For V not-irreducible, use induction $\square$

0.7. Cartan subalgebras, roots, coroots and $\mathfrak{sl}_2$'s

Let $\mathfrak{g}$ be a semisimple Lie algebra.

Def A: We call $\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra,

if (1) $\mathfrak{h}$ is a maximal Abelian subalgebra

(2) $\mathrm{ad}(h)$ is diagonalizable $\forall h \in \mathfrak{h}$ $\square$

Thm: All Cartan subalgebras are conjugate wrt. $\mathrm{Int}(\mathfrak{g})$.

Proof: Omitted $\square$

Example: $\mathfrak{h} = \mathfrak{sl}_n \cap \mathrm{diag} \subset \mathfrak{sl}_n$ is a Cartan subalgebra. $\square$

From now, choose a Cartan $\mathfrak{h} \subset \mathfrak{g}$

<u>Def B</u>: (1) For a rep. V of of $\mathfrak{h}$ and $\lambda \in \mathfrak{h}^*$,

we call

$$V_\lambda = \{ v \in V \mid hv = \lambda(h) v \quad \forall h \in \mathfrak{h} \}$$

the <u>weight space</u> of V (w.r.t. λ). If $V_\lambda \neq 0$,

we call λ a <u>weight</u> of V

(2) We call $0 \neq \alpha \in \mathfrak{h}^*$ a <u>root</u> of $\mathfrak{g}$ if it

is a weight of the adjoint rep. $\mathfrak{h} \to \mathfrak{gl}(\mathfrak{g})$

and denote by $\Delta \subset \mathfrak{h}^*$ the set of roots

(3) We denote by

$$\mathfrak{g} = \underset{\substack{\| \\ \mathfrak{h}}}{\mathfrak{g}_0} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$$

the weight space decomposition of $\mathfrak{g}$ $\square$

Exercise: Compute this for $\mathfrak{sl}_n$ (MUST DO) $\square$

For $\alpha, \beta \in \Delta$, $X \in \mathfrak{g}_\alpha$, $Y \in \mathfrak{g}_{-\alpha}$ and $H \in \mathfrak{h}$, we get

$$[H,[X,Y]] = (\alpha+\beta)(H)[X,Y] \quad \text{and}$$

$$([X,Y],H) = -\alpha(H)(X,Y) = \beta(H)(X,Y)$$

Some immediate consequences:

Lemma: Let $\alpha, \beta \in \Delta$.

(1) $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$

(2) $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subset \mathfrak{h}$

(3) $(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$ for $\beta \neq -\alpha$

(4) $(\mathfrak{g}_\alpha, \mathfrak{h}) = 0$

(5) $(\ ,\ )_{\mathfrak{h}\times\mathfrak{h}}$ is non-deg.

(6) $(\ ,\ )_{\mathfrak{g}_\alpha \times \mathfrak{g}_{-\alpha}}$ is non-deg pairing

(6) $\dim \mathfrak{g}_\alpha = \dim \mathfrak{g}_{-\alpha}$

(7) $\bigcap_{\alpha\in\Delta} \ker\alpha \subset \mathfrak{z}(\mathfrak{g}) = 0$

(8) $\mathfrak{h}^* = \langle \alpha \in \Delta \rangle_{\mathbb{C}}$ $\square$

Our next goal is to construct for each $\alpha \in \Delta$

a copy $s_\alpha \subset \mathfrak{g}$ of sl_2 with a basis $\{\underset{\mathfrak{g}_\alpha}{\underset{\cap}{e_\alpha}}, h_\alpha, \underset{\mathfrak{g}_{-\alpha}}{\underset{\cap}{f_\alpha}}\}$.

For this

$$V_{\alpha,\beta} = \bigoplus \mathfrak{g}_{\alpha+i\beta}$$

With a bit more work, one can show:

<u>Thm B</u>: For each $\alpha \in \Delta$, let

$$\mathfrak{s}_\alpha = \mathfrak{g}_\alpha \oplus [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \oplus \mathfrak{g}_{-\alpha}.$$

Then there is an isomorphism $\mathfrak{sl}_2 \cong \mathfrak{s}_\alpha$ and an $\mathfrak{sl}_2$-triple

$e_\alpha, h_\alpha = \alpha^\vee, f_\alpha \in \mathfrak{s}_\alpha$, s.t., $\mathfrak{g}_\alpha = \mathbb{C} e_\alpha$, $\mathfrak{g}_{-\alpha} = \mathbb{C} e_{-\alpha}$.

Moreover, h_α is uniquely determined by $\alpha(h_\alpha) = 2$.

<u>Proof</u>: Sketch: Pick $X_\alpha \in \mathfrak{g}_\alpha$, $X_\alpha \in \mathfrak{g}_{-\alpha}$, s.t.

$(X_\alpha, X_{-\alpha}) = 1$. Let $H_\alpha = [X_\alpha, X_{-\alpha}]$. Show

that $\alpha(H_\alpha) \neq 0$ and set $h_\alpha = \frac{2}{\alpha(H_\alpha)} H_\alpha$, $e_\alpha = X_\alpha$

and $f_\alpha = \frac{2}{\alpha(H_\alpha)} X_{-\alpha}$ $\square$

Def B: The elements $\alpha^\vee = h_\alpha \in \mathfrak{h}$ are called **coroots**. The set of coroots is denoted by

$$\Delta^\vee = \{\alpha^\vee \in \mathfrak{h}^* \mid \alpha \in \Delta\} \qquad \square$$

Lecture 4

0.9. Adjoint action of $\mathfrak{s}_\alpha$ on $\mathfrak{g}$.

For roots $\alpha, \beta \in \Delta$, we consider

$$V_{\alpha,\beta} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_{\beta + i\alpha} \subset \mathfrak{g}.$$

Now $V_{\alpha,\beta}$ is a f.d. representation of $\mathfrak{s}_\alpha \cong \mathfrak{sl}_2$

Let p, q be minimal, resp. maximal, s.t.

$\mathfrak{g}_{\beta + p\alpha} \neq 0$ and $\mathfrak{g}_{\beta + q\alpha} \neq 0$.

In particular $p \leq 0 \leq q$

Prop A: (1) $p+q = -\beta(h_\alpha) \in \mathbb{Z}$

(2) $\{\beta + j\alpha \mid j \in \mathbb{Z}\} \cap (\Delta \cup \{0\}) = \{\beta + j\alpha \mid p \leq j \leq q\}$

(3) $\beta + (p+q)\alpha = \beta - \beta(h_\alpha) \in \Delta$

(4) If $\alpha \pm \beta \neq 0$, then

(a) if $\alpha + \beta \notin \Delta$, $\beta(h_\alpha) \geq 0$

(b) if $\alpha - \beta \notin \Delta$, $\beta(h_\alpha) \leq 0$

(5) $[s_\alpha, s_\beta] = 0 \iff \beta(h_\alpha) = 0 \iff \alpha(h_\beta) = 0$

(6) If $\alpha + \beta \neq 0$, $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$

(7) $\mathbb{Z}\alpha \cap \Delta = \{\pm\alpha\}$

Proof: (1), (2), (3) $h_\alpha \in \mathfrak{s}_\alpha$ acts on $\mathfrak{g}_{\beta+i\alpha}$ via

$$(\beta+i\alpha)(h_\alpha) = \beta(h_\alpha) + 2i.$$

So, we obtain a $\mathfrak{s}_\alpha \cong \mathfrak{sl}_2$ rep. with h_α action

$$\begin{array}{ccccc} \mathfrak{g}_{\beta+p\alpha} & \cdots & \mathfrak{g}_{\beta+i\alpha} & \cdots & \mathfrak{g}_{\beta+q\alpha} \\ \circlearrowright & & & & \circlearrowright \\ \beta(h_\alpha)+2p & & & & \beta(h_\alpha)+2q \end{array}$$

From f.d. rep. th. of $\mathfrak{sl}_2$, see 0.4, we know that the extremal weights differ by a sign, so that

$$\beta(h_\alpha) + 2p = -(\beta(h_\alpha) + 2q)$$

and hence

$$\beta(h_\alpha) = -(p+q) \in \mathbb{Z}.$$

Moreover, $g_{\beta+i\alpha} \neq 0$ for all $p \leq i \leq q$.

In particular for $i = p+q$.

(4) - (7) : Exercise $\square$

Def: We call the sequence

$$\beta + p\alpha, \; \ldots\ldots, \; \beta + q\alpha$$

the α-string through β $\square$

0.10. Root system and Cartan matrix

Let $\mathfrak{h} \subset \mathfrak{g}$ be a semisimple Lie algebra with Cartan.

So far, we defined

(1) Roots $\quad \alpha \in \Delta \subset \mathfrak{h}^*$

(2) Coroots $\quad h_\alpha = \alpha^\vee \in \Delta^\vee \subset \mathfrak{h}$

(3) A bijection $\quad \Delta \to \Delta^\vee, \quad \alpha \mapsto \alpha^\vee$

We denote the natural pairing

$$\langle\,,\,\rangle : \mathfrak{h} \times \mathfrak{h}^* \to \mathbb{C}.$$

and for $R \subset \mathbb{C}$ a subring

$$\mathfrak{h}_R = \sum_{\alpha^\vee \in \Delta^\vee} A\alpha^\vee, \quad \mathfrak{h}^*_R = \sum_{\alpha \in \Delta} A\alpha$$

By 0.9. Prop (1), we have

$$\langle \beta, \alpha^\vee \rangle = \beta(h_\alpha) \in \mathbb{Z} \qquad \forall \alpha, \beta \in \Delta$$

So the pairing descends, for $R \subset \mathbb{C}$, to

$$\langle \, , \, \rangle : \mathfrak{h}_R \times \mathfrak{h}^*_R \to R.$$

We now collect all the properties of roots and coroots in a convenient framework.

Def: An abstract root system is a tuple

$(V, \Delta, \Delta^\vee, (\ )^\vee)$, such that

(1) V is a real vector space, generated by Δ.

(2) $\Delta \subset V$, $\Delta^\vee \subset V^\vee$ are finite subsets and

$(\ )^\vee$: — $^\vee$ a bijection

(3) For $\alpha, \beta \in \Delta$, $\langle \alpha, \beta^\vee \rangle \in \mathbb{Z}$ and $\langle \alpha, \alpha^\vee \rangle = 2$

(4) $r_\alpha : V \to V$, $v \mapsto v - \langle v, \alpha^\vee \rangle \alpha$

permutes Δ .

(5) If α and $c\alpha \in \Delta$, then $c = \pm 1$

Thm A $(\mathfrak{h}^*_{\mathbb{R}}, \Delta, \Delta^\vee, (\ )^\vee)$ is an abstract root system

Proof: (1) (2) Clear

(3) 0.7 Thm B + 0.9 Prop (1)

(4) 0.9 Prop (3)

(5) 0.9 Prop (7) $\square$

Remark:

In particular, there is a notion of positive, simple roots, Weyl group, ... ($\rightsquigarrow$ Bourbaki Lie Groups and Lie Algebras - Chapters 4-6) :

(1) Choose $\gamma \in h_{\mathbb{R}}$, s.t., $\langle \alpha, \lambda \rangle \neq 0 \quad \forall \alpha \in \Delta$.

Then define $\Delta_{\pm} = \{\alpha \in \Delta \mid \pm\langle \alpha, \gamma \rangle \geq 0\}$

and $\Delta_{\pm}^{\vee} = (\Delta_{\pm})^{\vee}$. The elents in $\Delta_{\pm}$ are

called positive/negative roots.

(2) A root $\alpha \in \Delta_{+}$ is called simple if it is

indecomposable, that is,

$$\alpha \cap \sum_{\substack{\beta \in \Delta^{+} \\ \beta \neq \alpha}} \mathbb{Z}_{\geq 0} \beta = \emptyset$$

The set of simple roots is denoted by Π.

Simple roots form a basis of h^{*}

(3) The <u>Weyl group</u> is defined by

$$W = \langle r_\alpha \mid \alpha \in \Delta \rangle = \langle r_\alpha \mid \alpha \in \Pi \rangle \subset GL(\mathfrak{h}_{\mathbb{Z}}^*)$$

. . . . $\square$

We will now take a closer look at root strings:

<u>Thm B</u>: Let $\alpha, \beta \in \Pi$ be simple roots. Then

(1) The α-string through β is of the form

$$\beta, \beta+\alpha, \dots, \beta - \langle \beta, \alpha^\vee \rangle \alpha$$

and $\langle \beta, \alpha^\vee \rangle \leq 0$

(2) $U(s_\alpha)\, \mathfrak{g}_\alpha$ is the f.d. sl_2-rep. of highest weight $-\langle \beta, \alpha^\vee \rangle$ and dimension $1 - \langle \beta, \alpha^\vee \rangle$

in particular $\mathrm{ad}(\mathfrak{g}_\alpha)^{1-\langle\beta,\alpha^\vee\rangle}\mathfrak{g}_\beta = 0$.

Proof: (1) Claim: $\alpha-\beta\notin\Delta$:

Assume $\alpha-\beta=\gamma\in\Delta$. If $\gamma\in\Delta^+$, $\beta+\gamma=\alpha$ $\lightning$ since α indecomposable. If $\gamma\in\Delta^-$, $-\gamma\in\Delta^+$ and $\alpha+(-\gamma)=\beta$ $\lightning$ since β indecomposable //claim.

Hence, the root string is of the form

$$\beta+p\alpha, \dots, \beta+q\alpha$$

with $p=0$. By 0.9 Prop (1), $q = p+q = -\beta(h_\alpha) = -\langle\beta,\alpha^\vee\rangle$. Moreover $-\langle\beta,\alpha^\vee\rangle = q\geq 0$.

(2) Exercise in $\mathfrak{sl}_2$-rep. theory $\square$

0.11 The Killing form on the Cartan.

Recall the proof of 0.7. Thm B. For $\alpha \in \Delta$, we picked $X_\alpha \in \mathfrak{g}_\alpha$, $X_{-\alpha} \in \mathfrak{g}_{-\alpha}$, s.t. $(X_\alpha, X_{-\alpha}) = 1$ and set $H_\alpha = [X_\alpha, X_{-\alpha}]$. Then

$$(H_\alpha, H) = ([X_\alpha, X_{-\alpha}], H) = \alpha(H)\,(X_\alpha, X_{-\alpha}) = \alpha(H).$$

Moreover, $h_\alpha = \frac{2H_\alpha}{(H_\alpha, H_\alpha)}$

So, when identifying $\mathfrak{h}$ and $\mathfrak{h}^*$ via the non-deg. form $(\,,\,)$ α corresponds to H_α. Note that we can also transport $(\,,\,)$ to a form on $\mathfrak{h}^*$, such that

$$(\alpha, \beta) = (H_\alpha, H_\beta).$$

Prop: let $\alpha, \beta \in \Delta$

(1) $\langle \beta, \alpha^\vee \rangle = \dfrac{2(\beta,\alpha)}{(\alpha,\alpha)}$

(2) $(\alpha,\beta) \in \mathbb{Q}$

(3) $(\alpha,\alpha) > 0$

(4) $(\, , \,)_{\mathfrak{h}_{\mathbb{R}} \times \mathfrak{h}_{\mathbb{R}}}$ is positive definite.

Proof: (1) We have

$$\langle \beta, \alpha^\vee \rangle = \beta(h_\alpha) = \beta\left(\frac{2H_\alpha}{(H_\alpha,H_\alpha)}\right) = \frac{2\beta(H_\alpha)}{(H_\alpha,H_\alpha)} = \frac{2(\beta,\alpha)}{(\alpha,\alpha)}$$

(2) Consider the $\mathfrak{s}_\alpha$-module $V = V_{\alpha,\beta} = \bigoplus \mathfrak{g}_{\beta+i\alpha}$.

Since $H_\alpha = [X_\alpha, X_{-\alpha}]$, $tr_V(ad(H_\alpha)) = 0$. On the

other hand

(2)–(4) Consider for $H, H' \in \mathfrak{h}$

$$(H, H') = \mathrm{tr}_{\mathfrak{g}}(\mathrm{ad}(H)\,\mathrm{ad}(H')) = \sum_{\beta \in \Delta} \beta(H)\,\beta(H').$$

Then, for $H = H' = H_\alpha$, we get

$$(H_\alpha, H_\alpha) = \sum_{\beta \in \Delta} \beta(H_\alpha)^2$$

Hence, we obtain

$$1 = (H_\alpha, H_\alpha) \sum_{\beta \in \Delta} \left(\frac{\beta(H_\alpha)}{(H_\alpha, H_\alpha)} \right)^2$$

$$= (H_\alpha, H_\alpha) \underbrace{\sum \left(\frac{1}{2} \langle \beta, \alpha^\vee \rangle \right)^2}_{\in \mathbb{Q}_{>0}}$$

So $(H_\alpha, H_\alpha) = (\alpha, \alpha) \in \mathbb{Q}_{>0}$. Hence

$$(\alpha, \beta) = \frac{1}{2} \langle \beta, \alpha^\vee \rangle (\alpha, \alpha) \in \mathbb{Q}.$$

Moreover, we see that h_α is a rational multiple of H_α. Hence, for $\mathbb{Q} \subset R \subset \mathbb{C}$

$$\mathfrak{h}_R = \sum R h_\alpha = \sum R H_\alpha .$$

To see that $(\, , \,)_{\mathfrak{h}_\mathbb{R} \times \mathfrak{h}_\mathbb{R}}$ is pos. def., let $H \in \mathfrak{h}_\mathbb{R}$. Then, as above

$$(H, H) = \sum_{\beta \in \Delta} \underbrace{(\underbrace{\beta(H)}_{\in \mathbb{R}})^2}_{\mathbb{R}_{\geq 0}} \geq 0 .$$

Since $(\, , \,)$ is non-deg., the statement follows $\square$

Lecture 5

0.11 Cartan matrix, Dynkin diagram and Classification

<u>Defn</u>: An $n\times n$-matrix $A=(a_{ij})_{i,j=1,\dots,n}$ is called a <u>generalized Cartan matrix</u> (GCM) if

(C1) $a_{ii}=2$ for $1\leq i\leq n$

(C2) $a_{ij}\in\mathbb{Z}_{\leq 0}$ for $i\neq j$

(C3) $a_{ij}=0 \Rightarrow a_{ji}=0$

Moreover, it is called a <u>Cartan matrix</u> if

(C4) $A=DB$ for D diagonal, B positive definite

The <u>Dynkin diagram</u> associated to a GCM is a decorated graph with:

(D1) The vertex set is $1, \dots, n$

(D2) Connect $i \neq j$ with $\max\{|a_{ij}|, |a_{ji}|\}$ lines

(D3) If $i \neq j$ and $|a_{ij}| \geq 2$, add an arrow

$$\underset{i}{\circ} \Leftarrow \underset{j}{\circ}$$

If the Dynkin diagram is a connected, we refer to it as

(C5) A is indecomposable $\square$

<u>Example</u>: (1) Let $(V, \Delta, \Delta^\vee, (\,)^\vee)$ be an abstract root system. Let $\Pi = \{\alpha_1, \dots, \alpha_n\}$ be a choice of positive roots. Let $a_{ij} = \langle \alpha_j, \alpha_i^\vee \rangle$. Then $A = (a_{ij})_{1 \leq i, j \leq n}$ is a Cartan matrix: (C1)–(C3) are straightforward

to see. To see C(4), pick a W-invariant scalar product $(,)$ on V. Then one may show that

$$\langle \alpha, \beta^\vee \rangle = \frac{2(\alpha,\beta)}{(\beta,\beta)} \quad \forall \alpha,\beta \in \Delta.$$

Let $B = ((\alpha_i, \alpha_j))_{1 \le i,j \le n}$ be the Gram matrix of $(,)$.
and $D = \mathrm{diag}\left(\frac{2}{(\alpha_i,\alpha_i)}\right)_{1 \le i \le n}$. Then $A = BD$.

Choosing this, helps us to draw the root system in the Euclidian v.s. $(V, (,))$.

Vice versa a Cartan matrix yields a root system!

(2) Let $\mathfrak{g} \supset \mathfrak{h}$ be a s.s. Lie alg. with Cartan.

We saw in 0.10 Thm A, that $(\mathfrak{h}^*_{\mathbb{R}}, \Delta, \Delta^\vee, (\ )^\vee)$

yields a root system, and hence a Cartan matrix.

In this case, the scalar product $(\, , \,)$ can be chosen as the Killing form. $\square$

<u>Exercise</u>: Convince yourself, using the Killing form, that

$$(\alpha^\vee, \beta^\vee) = \sum_{\gamma \in \Delta} \langle \gamma, \alpha^\vee \rangle \langle \gamma, \beta^\vee \rangle$$

Dually,

$$(\alpha, \beta) = \sum_{\gamma \in \Delta} \langle \alpha, \gamma^\vee \rangle \langle \beta, \gamma^\vee \rangle$$

This is an alternative formula to obtain B from A. $\square$

Thm A: All indecomposable Dynkin diagrams are of the form

Classical

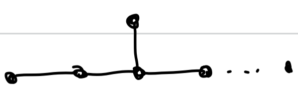
 A_n $\mathfrak{sl}_{n+1}$

 B_n $\mathfrak{so}_{2n+1}$

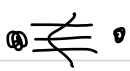
 C_n $\mathfrak{sp}_{2n}$

 D_n $\mathfrak{so}_{2n}$

Exceptional

 E_6, E_7, E_8

 F_4

 G_2

Proof: Omitted

Thm B: The are natural bijection between (isomorphism classes) of

(1) Cartan matrices (2) Dynkin diagrams

(3) (abstract) root systems (4) semisimple Lie algebras.

Proof: Omitted. $\square$

0.12. An extended example: Sp_4

Consider the symplectic matrix

$$J_4 = \begin{bmatrix} & & 1 & \\ & & & 1 \\ -1 & & & \\ & -1 & & \end{bmatrix}$$

and the associated symplectic Lie algebra

$$\mathfrak{g} = \mathfrak{sp}_4 = \{ X \in \mathfrak{gl}_4 \mid X J_4 + J_4 X^{tr} = 0 \}$$

$$= \left\{ \begin{bmatrix} A & B \\ C & -A^{tr} \end{bmatrix} \;\middle|\; \begin{array}{l} A, B, C \in \mathbb{C}^{2\times 2} \\ B = B^{tr},\ C = C^{tr} \end{array} \right\}$$

which has dimension $4 + 2 \cdot 3 = 10$. We choose the Cartan

$$\mathfrak{h} = \left\{ \begin{bmatrix} x_1 & & & \\ & x_2 & & \\ & & -x_1 & \\ & & & -x_2 \end{bmatrix} \right\}$$

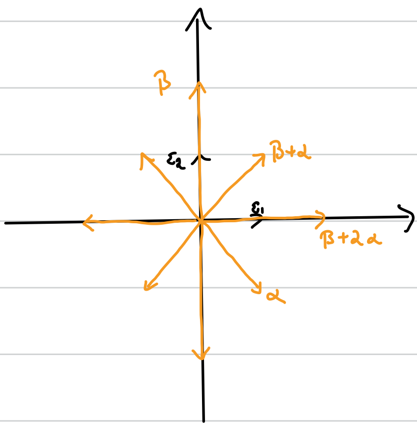
We denote the obvious bases of $\mathfrak{h}$ and $\mathfrak{h}^*$ by $\{\varepsilon_1, \varepsilon_2\}$ and $\{\varepsilon_1^*, \varepsilon_2^*\}$.

We then obtain the roots and coroots:

γ	$\mathfrak{g}_\gamma$	H_γ	$\gamma^\vee$
$\alpha = \varepsilon_1^* - \varepsilon_2^*$	$\left[\begin{array}{cc\|cc} & a & & \\ & & & \\ \hline & & & \\ & & -a & \end{array}\right]$	$\left[\begin{array}{cc\|cc} 1 & & & \\ & -1 & & \\ \hline & & -1 & \\ & & & 1 \end{array}\right]$	$\alpha^\vee = \varepsilon_1 - \varepsilon_2$
$\beta = 2\varepsilon_2^*$	$\left[\begin{array}{cc\|cc} & & & \\ & & & a \\ \hline & & & \\ & & & \end{array}\right]$	$\left[\begin{array}{cc\|cc} & & & \\ & 1 & & \\ \hline & & & \\ & & & -1 \end{array}\right]$	$\beta^\vee = \varepsilon_2$
$\alpha+\beta = \varepsilon_1^* + \varepsilon_2^*$	$\left[\begin{array}{cc\|cc} & & & a \\ & & a & \\ \hline & & & \\ & & & \end{array}\right]$	$\left[\begin{array}{cc\|cc} 1 & & & \\ & 1 & & \\ \hline & & -1 & \\ & & & -1 \end{array}\right]$	$(\alpha+\beta)^\vee = \varepsilon_1 + \varepsilon_2$
$2\alpha+\beta = 2\varepsilon_1^*$	$\left[\begin{array}{cc\|cc} & & a & \\ & & & \\ \hline & & & \\ & & & \end{array}\right]$	$\left[\begin{array}{cc\|cc} 1 & & & \\ & & & \\ \hline & & -1 & \\ & & & \end{array}\right]$	$(2\alpha+\beta)^\vee = \varepsilon_1$

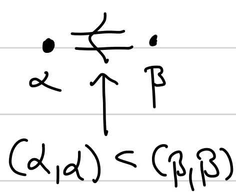
and $\mathfrak{g}_{-\gamma} = (\mathfrak{g}_\gamma)^{tr}$.

The root system has the form (visualised as Euclidian space):



and $\Pi = \{\alpha, \beta\}$ yields a choice of simple roots. The associated Cartan matrix and Dynkin diagram are:

$$A = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}$$



$A = DB$ for $D = \begin{pmatrix} 1/2 & \\ & 1 \end{pmatrix}$ $B = \begin{pmatrix} 4 & -2 \\ -2 & 2 \end{pmatrix}$

The α-string through β is of the form

$$\beta,\ \beta+\alpha,\ \underbrace{\beta+2\alpha}_{\substack{\| \\ -\langle\beta,\alpha^\vee\rangle}}$$

Correspondingly, we obtain an action of

$$\mathfrak{s}_\alpha = \left\langle e_\alpha = \left[\begin{array}{cc|cc} & 1 & & \\ & & & \\ \hline & & & \\ & & -1 & \end{array}\right],\ h_\alpha = \left[\begin{array}{cc|cc} 1 & & & \\ & -1 & & \\ \hline & & -1 & \\ & & & \end{array}\right],\ f_\alpha = e_\alpha^{tr} \right\rangle_{\mathbb{C}}$$

$$= \left\{ \begin{bmatrix} A & \\ & -A^{tr} \end{bmatrix} \middle| A \in \mathfrak{sl}_2 \right\} \cong \mathfrak{sl}_2$$

on the space

$$V_{\alpha,\beta} = \mathfrak{g}_\beta \oplus \mathfrak{g}_{\beta+\alpha} \oplus \mathfrak{g}_{\beta+2\alpha} = \left\{ \begin{bmatrix} & B \\ & \end{bmatrix} \middle| B = B^{tr} \right\}$$

The action is given by:

$$\left[\begin{bmatrix} A & \\ & -A^{tr} \end{bmatrix}, \begin{bmatrix} & B \\ & \end{bmatrix}\right] = \begin{bmatrix} & AB+BA^{tr} \\ & \end{bmatrix}.$$

This is exactly the sl_2-rep. $S^2(V) = L(-\langle\beta,\alpha^\vee\rangle) = L(2)$.

Similarly. $\langle\alpha+\beta,\alpha^\vee\rangle = 0$ corresponds to the fact that $[s_\alpha, s_{\alpha+\beta}] = 0$

Exercise: Do so_4 in the same detail.

Lecture 6

2 Kac-Moody Lie Algebras

$$\text{(C1)-(C3) GCM} \quad \longleftrightarrow \quad \text{Kac-Moody algebra}$$

$$\cup$$

$$\text{(C4)}_{\text{sym}} \quad A = \underset{\substack{\uparrow \\ \text{diag.}}}{D}\ \underset{\substack{\uparrow \\ \text{symmetric}}}{B} \quad \longleftrightarrow \quad \text{symmetrizable} \ \text{—"—}$$

$$\cup$$

$$\text{(C4)}_{\text{aff}} \quad A = \underset{\substack{\uparrow \\ \text{diag}}}{D}\ \underset{\substack{\uparrow \\ \text{pos. semidefinite} \\ \text{corank} \le 1}}{B} \quad \longleftrightarrow \quad \text{affine} \ \text{—"—}$$

$$\cup$$

$$\text{(C4) Cartan matrix} \quad \longleftrightarrow \quad \text{semisimple Lie algebra}$$

1.0. Some experiments

We will experiment and make sense of

$$A = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \quad \leftsquigarrow\!\rightsquigarrow \quad \hat{\mathfrak{sl}}_2$$

Let $\mathcal{L} = \mathbb{C}[t, t^{-1}]$ be the ring of Laurent polynomials.

We obtain the <u>loop algebra</u>

$$\mathcal{L}sl_2 = sl_2(\mathbb{C}[t,t^{-1}]) = \mathbb{C}[t,t^{-1}] \otimes sl_2$$

where for $x, y \in sl_2$, we have

$$[t^m \otimes x, t^n \otimes y] = t^{m+n}[x, y].$$

In particular $sl_2 = 1 \otimes sl_2 \subset \mathcal{L}sl_2$. If we take

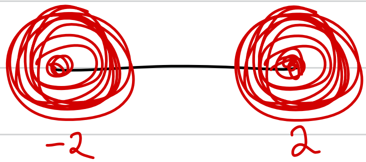
$\mathfrak{h}_0 = \langle h = 1 \otimes h \rangle_{\mathbb{C}}$ as "Cartan subalgebra", we obtain

a root space decomposition

$$\mathcal{L}sl_2 = \underset{-2}{\mathcal{L} \otimes f} \oplus \underset{0}{\mathfrak{h}} \oplus \underset{2}{\mathcal{L} \otimes e}$$

and for each $n \in \mathbb{Z}$ an sl_2-triple $t^n e, h, t^{-n} f$.

In our "root system", $2, -2$ appear with multiplicity ∞.



To distinguish the many copies of sl_2, need to make $\mathfrak{h}_0$ bigger. However, $\mathfrak{h}_0 \subset \mathcal{L}sl_2$ is already "maximal".

<u>Idea</u>: Make $\mathcal{L}sl_2$ bigger!

Define affine sl_2 as

$$\hat{\mathcal{L}}sl_2 = \mathcal{L}sl_2 \oplus \overset{\text{"central"}}{\overset{\Downarrow}{\mathbb{C}c}} \oplus \overset{\text{"derivation"}}{\overset{\Downarrow}{\mathbb{C}d}}$$

where the Lie bracket is defined by

$$[t^m \otimes x, t^n \otimes y] = t^{m+n}[x,y] + m\,\delta_{m+n,0}\,\overset{\text{trace form}}{(x,y)}\,c$$

$$[c, \hat{\mathcal{L}}sl_2] = 0$$

$$[d, t^n \otimes X] = n t^n \otimes X$$

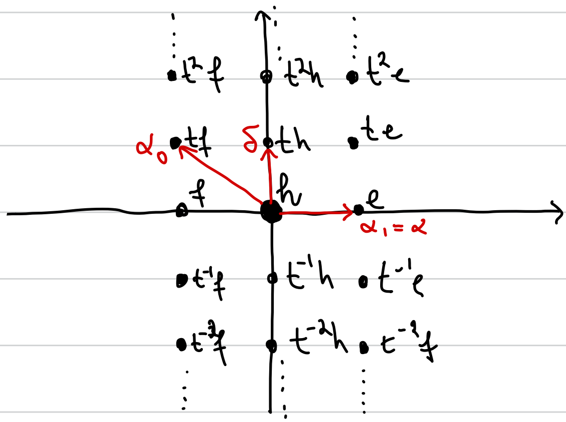
Then, we obtain the bigger Cartan:

$$\mathfrak{h} = \mathfrak{h}_0 \oplus \mathbb{C}c \oplus \mathbb{C}d = \langle \alpha_1^\vee = \alpha^\vee = h, c, d \rangle_{\mathbb{C}}$$

We write

$$\mathfrak{h}^* = \mathfrak{h}_0^* \oplus (\mathbb{C}c)^* \oplus (\mathbb{C}d)^* = \langle \alpha = \alpha_1, c^*, \delta = d^* \rangle_{\mathbb{C}}$$

We now obtain the root space decomposition



So we have roots of the form

$$\Delta = \{\pm\alpha + n\delta \mid n \in \mathbb{Z}\} \cup \{ n\delta \mid n \in \mathbb{Z}\setminus\{0\}\}.$$

We can also construct new coroots. For example, let $\alpha_0 = \delta - \alpha$. Then we get an $\mathfrak{sl}_2$-triple

$$e_0 = t \otimes f, \quad f_0 = t^{-1} \otimes e, \quad \alpha_0^\vee = h_0 = [e_0, f_0] = -h + c.$$

So, we have:

$\alpha_0 = \delta - \alpha$, $e_0 = t \otimes f$, $f_0 = t^{-1} \otimes e$, $\alpha_0^\vee = h_0 = -1 \otimes h + c$

$\alpha_1 = \alpha$, $e_1 = 1 \otimes e$, $f = 1 \otimes f$, $\alpha_1^\vee = h_1 = 1 \otimes h$

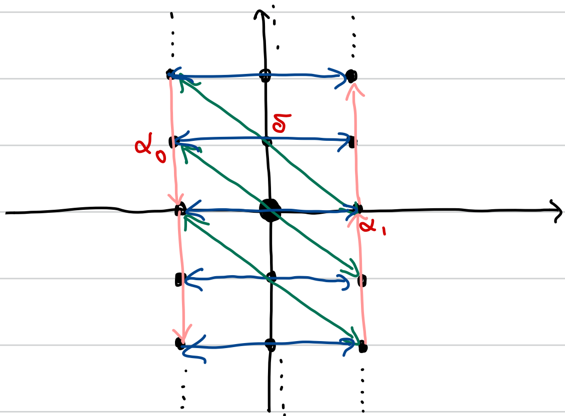
and obtain the generalized Cartan matrix

$$\left(\langle \alpha_j, \alpha_i^\vee \rangle\right)_{i=0,1} = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \qquad \underset{\alpha_0}{\bullet} \Longleftrightarrow \underset{\alpha_1}{\bullet}$$

In fact, e_0, e_1, f_1, f_2 and h generate $\hat{\mathcal{L}} sl_2$ and one can deduce from the Cartan matrix a system of generators!

We can also study the Weyl group

$$W = \langle s_0, s_1 \rangle$$



Note that $s_1 s_0 = \tau$ acts via the shear:

$$\alpha_1 \longmapsto \alpha_1 + 2\delta$$
$$\delta \longmapsto \delta$$

So that W is the infinite group

$$W = \langle s_0, s_1 \mid s_0^2 = s_1^2 = 1 \rangle$$

$$= \langle s_1, \tau \mid s_1^2 = 1,\ s_1 \tau s_1 = \tau^{-1} \rangle = \underset{\langle s_1 \rangle}{\underset{\shortparallel}{S_2}} \ltimes \underset{\langle \tau \rangle}{\underset{\shortparallel}{\mathbb{Z}}}$$

lets note some differences to the semisimple case

- The pairing $\langle , \rangle$ restricted to $\mathbb{Z}\Delta \times \mathbb{Z}\Delta^\vee$ is degenerate

- There is no scalar product $(,)$ on $\mathbb{R}\Delta$, s.t.

$$\langle \alpha, \beta^\vee \rangle = \frac{2(\alpha,\beta)}{(\beta,\beta)} \Rightarrow A \neq DB \text{ for } B \text{ pos. def.}$$

- There a roots, called <u>imaginary roots</u>,

$$\Delta^{im} = \{ n\delta \mid n \in \mathbb{Z} \setminus \{0\} \}$$

which are not <u>real</u>, that is, conjugate to simple roots

$$\Delta^{re} = W\Pi \neq \Delta = \Delta^{re} \uplus \Delta^{im}$$

<u>Exercise:</u> Experiment!

Lecture 7

1.1. Realizations

Our next goal is to construct $\mathfrak{g}$ from a GCM A.

We start with constructing $\mathfrak{h} \subset \mathfrak{g}$.

<u>Def</u>: A <u>realization</u> of a matrix $A \in \mathbb{C}^{n\times n}$ of rank ℓ is a triple $(\mathfrak{h}, \Pi, \Pi^\vee)$ where

- $\mathfrak{h}$ is a complex vector space,
- $\Pi = \{\alpha_1, \dots, \alpha_n\} \subset \mathfrak{h}^*$, $\Pi^\vee = \{\alpha_1^\vee, \dots, \alpha_n^\vee\} \subset \mathfrak{h}$

and the following conditions are fulfilled

(R1) Π and $\Pi^\vee$ are linearly independent

(R2) $\langle \alpha_i, \alpha_j^\vee \rangle = a_{ji}$

(R3) $n - \ell = \dim \mathfrak{h} - n$ $\square$

Prop: There is a natural bijection

$$\{\text{complex square matrices}\}/_{\substack{\text{permutation}\\ \text{of index set}}} \Leftrightarrow \{\text{realization}\}/_{\text{iso}} .$$

Proof: "$\Rightarrow$" Let $A \in \mathbb{C}^{n\times n}$ of rank l.

We can reorder the indices, such that

$$A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$$

and $A_1 \in \mathbb{C}^{l\times n}$ has full rank. Now, let

$$C = \begin{pmatrix} A_1 & 0 \\ A_2 & I_{n-l} \end{pmatrix},$$

$h = \mathbb{C}^{2n-l}$, $\alpha_1, \dots, \alpha_n$ the first n coordinate vectors in h

and $\alpha_1^\vee, \dots, \alpha_n^\vee$ the first n rows of C.

"$\Leftarrow$" Given a realization $(\mathfrak{h}, \Pi, \Pi^\vee)$, complete $\Pi = \{\alpha_1, \dots, \alpha_n\}$ to a basis $\{\alpha_1, \dots, \alpha_{2n-\ell}\}$ of $\mathfrak{h}^*$.

Then, we obtain

$$(\langle \alpha_j, \alpha_i^\vee \rangle) = \begin{array}{c|cc|c} & n & n-\ell & \\ \ell & A_1 & B & \ell \\ n-\ell & A_2 & D & n-\ell \end{array}$$

By adding linear combinations of $\{\alpha_1, \dots, \alpha_n\}$ to $\{\alpha_{n+1}, \dots, \alpha_{2n-\ell}\}$, we can assume $B = 0$.

Since D is invertible, we can replace $\{\alpha_{n+1}, \dots, \alpha_{2n-\ell}\}$ by a linear combination, so that $D = I$. Hence, the realisation is isomorphic to the one constructed in "$\Rightarrow$" $\square$

<u>Def B</u>: Let $(\mathfrak{h}, \Pi, \Pi^\vee)$ be a realisation. Then

$$Q = \sum_{i=1}^{n} \mathbb{Z}\alpha_i \quad \text{and} \quad Q^\vee = \sum_{i=1}^{n} \mathbb{Z}\alpha_i^\vee$$

are called the <u>root</u> and <u>coroot</u> <u>lattice</u>. Moreover, we define

$$Q_+ = \sum \mathbb{Z}_{\geq 0}\alpha_i \subset Q \; , \quad Q_+^\vee = \sum \mathbb{Z}_{\geq 0}\alpha_i^\vee$$

For $\alpha = \sum k_i \alpha_i$, the height is

$$\mathrm{ht}\,\alpha = \sum k_i$$

and for $\lambda, \mu \in \mathfrak{h}^*$, we write $\lambda \leq \mu$ if

$\mu - \lambda \in Q_+$ $\square$

<u>Example</u>: Let $A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$. Then $\operatorname{rank} A = 1$, so $\dim \mathfrak{h} = 2 \cdot n - \ell = 4 - 1 = 3$. The choice of $\mathfrak{h}$, Π and $\Pi^\vee$ as in 1.0 yields a realisation.

1.2 An auxiliary algebra

Let $A \in \mathbb{C}^{n \times n}$ with realization $(\mathfrak{h}, \Pi, \Pi^\vee)$.

<u>Def</u>: We define $\tilde{\mathfrak{g}}(A)$ as the Lie algebra with generators $e_i, f_i, \mathfrak{h}$ and relations

(R1) $\quad [e_i, f_j] = \delta_{ij} \alpha_i^\vee$

(R2) $\quad [h, h'] = 0 \quad \forall h, h' \in \mathfrak{h}$

(R3) $\quad [h, e_i] = \langle \alpha_i, h \rangle e_i$

(R4) $\quad [h, f_i] = -\langle \alpha_i, h \rangle f_i.$

We denote by $\tilde{n}_+$ (or $\tilde{n}_-$) the subalgebras generated by $e_1, \dots, e_n$ and $f_1, \dots, f_n$ $\quad \square$

Thm: We get

(1) $\tilde{\mathfrak{g}}(A) = \tilde{\mathfrak{n}}_- \oplus \mathfrak{h} \tilde{\oplus} \mathfrak{n}_+$

(2) $\tilde{\mathfrak{n}}_+$ and $\tilde{\mathfrak{n}}_-$ are freely generated by $e_1, \dots, e_n$ and $f_1, \dots, f_n$, respectively.

(3) There is a unique involution $\tilde{\omega}$ on $\mathfrak{g}(A)$, such that,

$$e_i \mapsto -f_i, \; f_i \mapsto -e_i, \; h \mapsto -h$$

(4) There is a root space decomposition

$$\tilde{\mathfrak{g}}(A) = \bigoplus_{\substack{\alpha \in Q_+ \\ \alpha \neq 0}} \mathfrak{g}_{-\alpha} \oplus \mathfrak{h} \oplus \bigoplus_{\substack{\alpha \in Q_+ \\ \alpha \neq 0}} \mathfrak{g}_\alpha$$

and $\dim \mathfrak{g}_\alpha < \infty$, $\tilde{\mathfrak{g}}_\alpha \subset \tilde{\mathfrak{n}}_\pm$ for $\alpha \in \pm Q_+ \setminus \{0\}$ $\square$

(5) There is a unique maximal $\mathfrak{r} \subset \mathfrak{g}(A)$, s.t. $\mathfrak{r}$ is and ideal and $\mathfrak{r} \cap \mathfrak{h} = 0$. It fulfills

$$\mathfrak{r} = (\mathfrak{r} \cap \mathfrak{n}^-) \oplus (\mathfrak{r} \cap \mathfrak{n}^-) \qquad \text{(as ideals)}$$

<u>Proof</u>: <u>Idea</u>: <u>If</u> $\mathfrak{n}^-$ is free, then $\mathcal{U}(\tilde{\mathfrak{n}}_-) = T^\bullet(V)$ for $V = \bigoplus \mathbb{C} e_i$. Then, we would get a Verma-module-like representation: $\tilde{M}(\lambda) = \mathcal{U}(\tilde{\mathfrak{g}}) \otimes_{\mathcal{U}(\mathfrak{h} \oplus \mathfrak{n}^+)} \mathbb{C}_\lambda$.

By the PBW-theorem (0.2 Thm) $\tilde{M}(\lambda) \cong \mathcal{U}(\tilde{\mathfrak{n}}_-) \cong T^\bullet(V)$ as vector spaces. So, we proceed in the opposite direction: We construct an action of $\tilde{\mathfrak{g}}(A)$ on $T^\bullet(V)$ mimicking $\tilde{M}(\lambda)$ // Idea.

Let $\lambda \in \mathfrak{h}^*$, $V = \bigoplus \mathbb{C} v_i$. We define an action of $\mathfrak{g}(A)$ on $T(V)$ via:

(a) $f_i(a) = v_i \otimes a$ for $a \in A$

(b) $h(1) = \langle \lambda, h \rangle 1$ and (inductively)

$$h(v_j \otimes a) = -\langle \alpha_j, h \rangle v_j \otimes a + v_j \otimes h(a), \quad a \in T(V)$$

(c) $e_i(1) = 0$ and

$$e_i(v_j \otimes a) = \delta_{ij} \alpha_i^\vee(a) + v_j \otimes e_i(a), \quad a \in T(V)$$

<u>Claim</u>: This defines a representation.

<u>Proof of claim</u>: Need to check relations (R1) – (R4):

(R1) $(e_i f_j - f_i e_i)(a) = e_i(v_j \otimes a) - v_j \otimes e_i(a)$

$$= \delta_{ij}\, \alpha_i^\vee(a) + v_j \otimes e_i(a) - v_j \otimes e_i(a)$$

$$= \delta_{ij}\, \alpha_i^\vee(a)$$

(R2) Check that $\mathfrak{h}$ acts diagonally by induction.

(R3) Exercise

(R4) $(hf_j - f_j h)(a) = h(v_j \otimes a) - v_j \otimes h(a)$

$$= -\langle a_j, h\rangle v_j \otimes a + v_j \otimes h(a) - v_j \otimes h(a)$$

$$= -\langle a_j, h\rangle f_j(a)$$

// Claim

(i) Using the relations, it is easy to see that $\mathfrak{g}(A) = \tilde{n}_- + \mathfrak{h} + \tilde{n}_+$.

Now, let $u = n_- + h + n_+ = 0$. Then, by acting on $|E|T(V)$, we get for all $\lambda \in \mathfrak{h}^*$

$$0 = u(1) = n_-(1) + \langle \lambda, h \rangle = 0 \quad .$$

Since $n_-(1)$ does not depend on λ, $\langle \lambda, h \rangle = 0$.

The map $\mathfrak{n}_- \to T(V)$, $n \mapsto n(1)$ is a map of Lie algebras $\mathfrak{n}_- \to (T(V), [\,,\,])$ and hence factors as $\mathfrak{n}_- \hookrightarrow \mathcal{U}(\mathfrak{n}_-) \xrightarrow{\varphi} T(V)$. Since $T(V)$ is a free, the inclusion $V \to \mathfrak{n}_-$ yields a map $T(V) \to \mathcal{U}(\mathfrak{n}^-)$ which is inverse to φ, so $\mathcal{U}(\mathfrak{n}_-) \cong T(V)$. Hence $\mathfrak{n}_- \to T(V)$ is injective and $n_-(1) = 0 \Rightarrow n_- = 0$. Hence (1) follows.

(2) By PBW, $\tilde{n}_-$ is freely generated by $f_1, \dots, f_n$.

(3) Clear, by checking (R1)–(R4).

(4) By (R3) and (R4)

$$\tilde{n}_{\pm} = \bigoplus_{\substack{\alpha \in Q_+ \\ \alpha \neq 0}} \tilde{g}_{\pm\alpha}$$

Moreover $\dim \tilde{g}_{\pm\alpha} \leq n^{|ht\alpha|}$

(5) We will show in 1.4 Prop that for any ideal $i \subset g$

$$i = \bigoplus_{\alpha} \tilde{g}_\alpha \cap i$$

Let $\mathfrak{r}' = \sum_{\substack{i \subset \mathfrak{g} \\ i \cap \mathfrak{h} = 0}} i = \sum_{\substack{i \subset \mathfrak{g} \\ i \cap \mathfrak{h} = 0 \\ \alpha \neq 0}} \tilde{\mathfrak{g}}_\alpha \cap i \subset \bigoplus_{\alpha \neq 0} \tilde{\mathfrak{g}}_\alpha$

So $\mathfrak{r}' \cap \mathfrak{h}$. It follows easily that $\mathfrak{r} = \mathfrak{r}'$ is the ideal we were looking for. By (1) and the claim, $\mathfrak{r} = \mathfrak{r} \cap \tilde{\mathfrak{n}}_- \oplus \mathfrak{r} \cap \tilde{\mathfrak{n}}_+$ as vector spaces. Now $[f_i, \mathfrak{r} \cap \tilde{\mathfrak{n}}_+] \subset \mathfrak{r} \cap (\mathfrak{h} \oplus \tilde{\mathfrak{n}}_+) = \mathfrak{r} \cap \tilde{\mathfrak{n}}_+$, and hence $[\tilde{\mathfrak{g}}(A), \mathfrak{r} \cap \tilde{\mathfrak{n}}_+] \subset \mathfrak{r} \cap \tilde{\mathfrak{n}}_+$. So $\mathfrak{r} \cap \tilde{\mathfrak{n}}_\pm \subset \tilde{\mathfrak{g}}(A)$ are ideals $\square$

Lecture 8

1.3 Kac-Moody algebra – Definition

Def: Let A be an $n\times n$-matrix with realization $(\mathfrak{h}, \Pi, \Pi^\vee)$. The Kac-Moody algebra with Cartan matrix A is defined as

$$\mathfrak{g}(A) = \tilde{\mathfrak{g}}(A)/\mathfrak{r}$$

where $\tilde{\mathfrak{g}}(A)$ and $\mathfrak{r}$ are defined as in 1.2 Def + Thm (4).

Then $(\mathfrak{g}(A), \mathfrak{h}, \Pi, \Pi^\vee)$ is the quadruple associated to A, $\mathfrak{h} \subset \mathfrak{g}(A)$ the Cartan subalgebra and $e_i, f_i \in \mathfrak{g}(A)$ the Chevalley generators. $\square$

Exercise: Work out the universal property of $(\mathfrak{g}(A), \mathfrak{h}, \Pi, \Pi^\vee)$ $\square$

We fix a quadruple $(\mathfrak{g}(A), \mathfrak{h}, \Pi, \Pi^\vee)$.

<u>Prop</u>: (1) There is a triangular/root space decomposition

$$\mathfrak{g}(A) = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+ = \bigoplus_{\alpha \in \Delta^-} \mathfrak{g}_\alpha \oplus \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$$

where $\Delta = \{\alpha \in Q \setminus \{0\} \mid \mathfrak{g}_\alpha \neq 0\}$ is the set of <u>roots</u>,

and $\Delta_\pm = \Delta \cap Q_\pm$ the set of <u>positive</u>/<u>negative</u> <u>roots</u>.

(2) $\mathfrak{g}_{\alpha_i} = \mathbb{C} e_i$, $\mathfrak{g}_{-\alpha_i} = \mathbb{C} f_i$ and $\mathfrak{g}_{s\alpha_i} = 0$ for $\alpha_i \in \Pi$, $s \neq \pm 1$

(3) If $\beta \in \Delta_+ \setminus \{\alpha_i\}$, then $(\beta + \mathbb{Z}\alpha_i) \cap \Delta \subset \Delta_+$.

(4) The Chevalley involution $\tilde{\omega}$ and $\tilde{\mathfrak{g}}(A)$ descends

to an involution ω on $\mathfrak{g}(A)$, and $\omega(\mathfrak{g}_\alpha) = \mathfrak{g}_{-\alpha}$.

Hence $\Delta_- = -\Delta_+$

(5) Let $\mathfrak{g}'(A) = [\mathfrak{g}(A), \mathfrak{g}(A)]$ Then

$\mathfrak{g}(A) = \mathfrak{g}'(A) + \mathfrak{h}$ and $\mathfrak{g}'(A) \cap \mathfrak{h} = \mathfrak{h}' = \sum \mathbb{C}\alpha_i^\vee$.

Moreover, $\mathfrak{g}'(A)$ is generated by the e_i, f_i's

and $\mathfrak{g}'(A) \cap \mathfrak{g}_\alpha = \mathfrak{g}_\alpha$ for all $\alpha \in \Delta$

(6) If $\mathfrak{c} \subset \mathfrak{g}(A)$ is an ideal with $\mathfrak{c} \cap \mathfrak{h} = 0$, Then $\mathfrak{c} = 0$

Proof: Exercise ∎

1.4 Gradings and the center

Def: Let M be an Abelian group. An <u>M-grading</u> on a vector space V is a decomposition $V = \bigoplus_{\alpha M} V_\alpha$.

A subspace $U \subset V$ is <u>graded</u> if $U = \bigoplus_{\alpha \in M} U \cap V_\alpha$.

An <u>M-grading</u> on a Lie algebra $\mathfrak{g} = \bigoplus_{\alpha \in M} \mathfrak{g}_\alpha$ is a grading, s.t., $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$. $\square$

Prop: Let $\mathfrak{h}$ be commutative and V a diagonizable $\mathfrak{h}$-module. Then the weight space decomposition

$$V = \bigoplus_{\alpha \in \mathfrak{h}^*} V_\alpha$$

is an $\mathfrak{h}^*$-grading. If $W \subset V$ is a submodule, then

W is graded.

<u>Proof</u>: Write $w \in W$ as $w = \sum_{i=0}^{m} w_i$, where

$\{\lambda_0, \dots, \lambda_m\} \subset \mathfrak{h}^*$ is a finite set and $w_i \in V_{\lambda_i}$

Choose $h \in \mathfrak{h}$, s.t., $\langle \lambda_i, h \rangle \neq \langle \lambda_j, h \rangle$ for $i \neq j$.

Then we obtain

$$h^j(w) = \sum_{\lambda \in S} \langle \lambda_i, h \rangle^j \, w_i \in W$$

Let $X = (\langle \lambda_i, h \rangle^j)_{i,j=0,\dots,m}$ Then X is

a Vandermonde matrix and hence invertible, and

$$X^{-1}(h^i(w))_i = (w_i)_i$$

Hence $w_i \in W$ $\square$

Example: Consider the Kac-Moody algebra $\mathfrak{g}(A)$.

Then $\mathfrak{g}(A)$ is clearly Q-graded.

For any functional $s: Q \to \mathbb{Z}$, we obtain a $\mathbb{Z}$-grading, via

$$\mathfrak{g} = \bigoplus_{n \in \mathbb{Z}} \mathfrak{g}_n(s), \text{ where } \mathfrak{g}_n = \bigoplus_{s(\alpha)=n} \mathfrak{g}_\alpha .$$

For example, the principal grading on $\mathfrak{g}(A)$ arises from $|\,|: Q \to \mathbb{Z}$, $\alpha_i \mapsto 1$. Then

$$\mathfrak{g}_n(|\,|) = \bigoplus_{\mathrm{ht}\,\alpha = n} \mathfrak{g}_\alpha$$

and, in particular,

$$\mathfrak{g}_0(|\,|) = \mathfrak{h}, \quad \mathfrak{g}_1(|\,|) = \bigoplus \mathbb{C} e_i, \quad \mathfrak{g}_{-1}(|\,|) = \bigoplus \mathbb{C} f_i$$

and $\mathfrak{n}_{\pm} = \bigoplus_{n \geq 1} \mathfrak{g}_{\pm n}(11)$ ▯

<u>Lemma</u>: Let $a \in \mathfrak{n}_+$ such $[a, f_i] = 0$ for all $i = 1, \dots, n$.

Then $a = 0$. (Similar for $a \in \mathfrak{n}^-$)

<u>Proof</u>: Let $a \in \mathfrak{n}_+$, s.t. $[a, f_i] = 0$ $\forall a$. Then

$[a, \mathfrak{g}_{-1}(11)] = 0$. Let $V = \sum_{i,j} (\operatorname{ad} \mathfrak{g}_1(11))^i (\operatorname{ad} \mathfrak{h})^j a$.

Then V is clearly invariant under $\operatorname{ad} \mathfrak{g}_1(11)$ and

$\operatorname{ad} \mathfrak{h}$. Similarly it is $\operatorname{ad} \mathfrak{g}_{-1}(11)$ invariant using $\operatorname{ad} \mathfrak{g}_{-1}(4)\, a = 0$.

Hence, $V \subset \mathfrak{n}^+$ is an ideal. By 1.3 Prop (6) $V = 0 \Rightarrow a = 0$ ▯

Thm: (1) The center of $\mathfrak{g}(A)$ and $\mathfrak{g}'(A)$ is

$$\mathfrak{z} = \{ h \in \mathfrak{h} \mid \langle \alpha_i, h \rangle = 0 \ \forall i = 1, \dots, n \} = \bigcap_{\alpha \in \Pi} \ker \alpha$$

and $\dim \mathfrak{z} = n - \ell$

(2) let $I_1, I_2 \subset \{1, \dots, n\}$, $I_1 \cap I_2 = \emptyset$ and $a_{ij} = a_{ji} = 0$

let $\beta_{1,2} \in \sum_{\alpha \in I_{1,2}} \mathbb{Z}\alpha$. If $\alpha = \beta_1 + \beta_2 \in \Delta$, then either

β_1 or β_2 are zero

(3) $\mathfrak{g}(A)$ is simple if and only if $\det(A) \neq 0$ and

the Dynkin diagram is connected

(4) If the Dynkin diagram is connected, then for

every ideal $i \subset \mathfrak{g}$ either $i \subset \mathfrak{z}$ or $\mathfrak{g}'(A) \subset i$.

Proof (1) "$\subseteq$" Let $z \in \mathfrak{z}$. Write $z = \sum_{i \in \mathbb{Z}} z_i$ for

$z_i \in \mathfrak{z} \cap \mathfrak{g}_i(\mathbb{1})$. If $i > 0$, $z_i \in \mathfrak{n}^+$.

Since $[z_i, \mathfrak{g}_{-1}(\mathbb{1})] = 0$, $z_i = 0$. Similarly, $z_i = 0$

for $i < 0$, so $z = z_0 \in \mathfrak{g}_0(\mathbb{1}) = \mathfrak{h}$.

Now $[c, e_i] = \langle \alpha_i, c \rangle e_i = 0$, so $c \in \ker \alpha_i$,

for all i.

"$\supseteq$" Let $z \in \bigcap \ker \alpha_i$. Then $[z, e_i] = [z, f_i] = 0$, so

$z \in \mathfrak{z}$.

Since the α_i are linearly independent, $\dim \mathfrak{z} = \dim \mathfrak{h} - n$

$= n - \ell$. Now $\dim\left(\bigcap \ker \alpha_i \cap \sum \mathbb{C} \alpha_i^\vee\right) = \operatorname{corank} A = n - \ell$.

It follows that $\mathfrak{z} \cap \mathfrak{h}' = \mathfrak{z}$.

(2) Let $i \in I_1$, $j \in I_2$. Then

$$[[e_i, e_j], f_k] = [[e_i, f_k], e_j] + [e_i, [e_j, f_k]]$$

$$= \delta_{i,k}[\alpha_i^\vee, e_j] + [e_i, \delta_{jk}\alpha_j^\vee]$$

$$= \delta_{ik}\langle \alpha_j, \alpha_i^\vee \rangle - \delta_{jk}\langle \alpha_i, \alpha_j^\vee \rangle = 0.$$

Hence, by the Lemma, $[e_i, e_j] = 0$. Similarly $[f_i, f_j] = 0$. Let $\mathfrak{g}^{(1,2)}$ the subalgebras generated by e_i, f_i for $i \in I_{1,2}$. Then $[\mathfrak{g}^{(1)}, \mathfrak{g}^{(2)}] = 0$.

Now $\mathfrak{g}_\alpha \subset \langle \mathfrak{g}^{(1)}, \mathfrak{g}^{(2)} \rangle = \mathfrak{g}^{(1)} \times \mathfrak{g}^{(2)}$, so either $\mathfrak{g}_\alpha \in \mathfrak{g}^{(1)}$ or $\mathfrak{g}_\alpha \in \mathfrak{g}^{(2)}$

(3) "$\Rightarrow$" Exercise.

"$\Leftarrow$" Let $\mathfrak{i} \subset \mathfrak{g}(A)$ be a nontrivial ideal. Then $\mathfrak{i} \cap \mathfrak{h} \neq 0$ by 1.3 Prop (6). Let $0 \neq h \in \mathfrak{i} \cap \mathfrak{h}$. Since $\det A \neq 0$, $\mathfrak{z} = \bigcap \ker \alpha_i = 0$. Hence, there is a j, such that $[h, e_j] = a e_j \neq 0$. So $e_j \in \mathfrak{i}$ and $\alpha_j^\vee = [e_j, f_j] \in \mathfrak{i}$. Using a path through the Dynkin diagram, we see that $e_i, f_i, \alpha_i^\vee \in \mathfrak{i}$ for all i. Since $\det A \neq 0$, $\mathfrak{h}' = \mathfrak{h}$. So $\mathfrak{i} = \mathfrak{g}(A)$

(4) Exercise $\square$

Lecture 9

<u>Example</u>: Let $A = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$ such that

$$\mathfrak{g}(A) = \tilde{\mathcal{L}}\mathfrak{sl}_2 = \mathcal{L}\mathfrak{sl}_2 \oplus \mathbb{C}c \oplus \mathbb{C}d$$

Recall that

$$\mathfrak{h} = \mathfrak{h}_0 \oplus \mathbb{C}c \oplus \mathbb{C}d = \langle h, c, d \rangle_{\mathbb{C}}$$

$$\mathfrak{h}^* = \mathfrak{h}_0^* \oplus \mathbb{C}c^* \oplus \mathbb{C}d^* = \langle \alpha, c^*, d^* \rangle_{\mathbb{C}}$$

Moreover $\Pi = \{\alpha_0 = d^* - \alpha,\ \alpha_1 = \alpha\}$, $\Pi^\vee = \{\alpha_0^\vee = -h + c,\ \alpha_1^\vee = h\}$.

Hence, we obtain

$$\mathfrak{h}' = \mathfrak{g}'(A) \cap \mathfrak{h} = \mathbb{C}\alpha_0^\vee \oplus \mathbb{C}\alpha_1^\vee = \langle h, c \rangle_{\mathbb{C}},$$

$$\mathfrak{g}'(A) = \mathcal{L}\mathfrak{sl}_2 \oplus \mathbb{C}c \quad \text{and}$$

$$\mathfrak{z} = \ker \alpha_0 \cap \ker \alpha_1 = \ker \alpha \cap \ker d^* = \mathbb{C}c \qquad \square$$

1.5 Symmetrization

Our goal is to construct a Casimir operator Ω.

For this, we will need a bilinear form similar to the Killing form. This is possible in the following case:

<u>Def</u> A Let A be an $n \times n$-matrix.

Then A is called <u>symmetrizable</u> if $A = DB$ for $D = \operatorname{diag}(\varepsilon_i)$ diagonal and invertible and B symmetric.

In this case, B is called a <u>symmetrization</u>. $\square$

From now let $A = DB$ with fixed decomposition, where $D = \operatorname{diag}(\varepsilon_i)$. Moreover, let $(\mathfrak{h}, \Pi, \Pi^\vee)$ be a realization.

Then, we can define a symmetric bilinear forms on

$\mathfrak{h}$ and $\mathfrak{h}^*$ in the following way:

Choose a complement $\mathfrak{h} = \mathfrak{h}' \oplus \mathfrak{h}''$. Set

$$(\alpha_i^\vee, h) = \langle \alpha_i, h \rangle \varepsilon_i$$

$$(h, h'') = 0 \quad \text{for } h', h'' \in \mathfrak{h}''$$

<u>Lemma</u>: (1) $(,)$ is well-defined and symmetric

(2) $\ker\big((- , -) : \mathfrak{h}' \to \mathfrak{h}^* \big) = \mathfrak{z}$

(3) $(,)$ is non-degenerate

<u>Proof</u>: (1) Use that $\Pi^\vee$ is linearly independent and

$(\alpha_i^\vee, \alpha_j^\vee) = b_{ij} \varepsilon_i \varepsilon_j$

(2) $\ker((-,-): \mathfrak{h}' \to (\mathfrak{h}')^*) = \bigcap_i \ker((\alpha_i^\vee, -))$

$= \bigcap_i \ker(\alpha_i) = \mathfrak{z}$

(3) Exercise $\square$

Since $(,)$ is non-degenerate, it yields an iso.:

$\nu: \mathfrak{h} \to \mathfrak{h}^*, \quad h \mapsto (h, -)$

and a form $(,)$ on $\mathfrak{h}^*$, such that $\nu(\alpha_i^\vee) = \varepsilon_i \alpha_i$

and $(\alpha_i, \alpha_j) = b_{ij} = a_{ij}/\varepsilon_i$.

<u>Thm</u>: We can extend $(-,-)$ to a non-deg.

bilinear form on $\mathfrak{g}(A)$, s.t.,

(a) $(-,-)$ is $\mathfrak{g}(A)$-invariant

(b) $(-,-)|_{\mathfrak{h}}$ is as defined above

(c) $(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$ if $\alpha \neq -\beta$

(d) $(\cdot|\cdot)|_{\mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha}}$ is non degenerate

(e) $[x,y] = (x,y)\, \nu^{-1}(\alpha)$ for $x \in \mathfrak{g}_\alpha$, $y \in \mathfrak{g}_{-\alpha}$

<u>Proof Sketch</u>: We extend $(\,,\,)$ on $\mathfrak{g}(N) = \bigoplus_{|i| \leq N} \mathfrak{g}_i$ (1).

For $N=0$, $\mathfrak{g}(N) = \mathfrak{h}$ and we take the form defined above.

For $\mathfrak{g}(1)$, we set

$$(e_i, f_j) = \delta_{ij}\, \varepsilon_i \quad \text{and}$$

$$(\mathfrak{g}(0), \mathfrak{g}(\pm 1)),\ (\mathfrak{g}(\pm 1), \mathfrak{g}(\pm 1)) = 0$$

Then, one extends using the invariance property.

To see that $(\,,\,)$ is non.deg., we use that its kernel is an ideal not intersecting $\mathfrak{h}$ $\square$

<u>Remark</u>: A symmetrizable generalized Cartan matrix $A = DB$ can always be factored such that $D \in \mathbb{Q}_{>0}$.

If A is indecomposable, D is unique up to isomorphism.

Then $(\alpha_i, \alpha_i) > 0$, $(\alpha_i, \alpha_j) \leq 0$ for $i \neq j$, $\alpha_i^\vee = \frac{2}{(\alpha_i, \alpha_i)} \nu^{-1}(\alpha_i)$

and $A = \left(\frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)} \right)_{i,j=1,\dots,n}$ as in the semisimple case $\square$

Example: For $\hat{\mathcal{L}}sl_2 = \mathcal{L}sl_2 \oplus \mathbb{C}c \oplus \mathbb{C}d$, $A = I_2 A$ is already symmetric. Recall that $\Pi = \{\alpha_0 = d^* - \alpha,\ \alpha_1 = \alpha\}$, $\Pi^\vee = \{\alpha_0^\vee = -h + c,\ \alpha_1^\vee = h\}$

Hence, $\mathfrak{h} = \mathfrak{h}' \oplus \mathfrak{h}'' = \langle h, c\rangle \oplus \langle d\rangle$ and

$$(\,,\,)_{\mathfrak{h}\times\mathfrak{h}} = \begin{array}{c|ccc} & h & c & d \\ \hline h & 2 & 0 & 0 \\ c & 0 & 0 & 1 \\ d & 0 & 1 & 0 \end{array}$$

so that $(\,,\,)_{\mathfrak{h}'\times\mathfrak{h}'} = \begin{array}{cc} h & c \\ \end{array}\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$, so that $\ker(\,,\,)_{\mathfrak{h}'\times\mathfrak{h}'} = \mathbb{C}c = \mathfrak{z}$.

We now extend $(\,,\,)$ to $\hat{\mathcal{L}}sl_2$. We want

$(e_i, f_j) = \delta_{ij}\varepsilon_i = \delta_{ij}$, so that $(e_0, f_0) = (tf, t^{-1}e) = 1$

and $(e_1, f_1) = (e, f) = 1$.

To compute other values, we use the $\mathfrak{g}$-invariance. For example

$th = [e_1, e_0]$ and $t^{-1}h = [f_0, f_1]$, so that

$$(th, t^{-1}h) = ([e_1, e_0], [f_0, f_1]) = (e_1, [e_0, [f_0, f_1]])$$

$$= (e_1, [[e_0, f_0], f_1] + [f_0, [e_0, f_1]]) = (e_1, [\alpha_0^\vee, f_1])$$

$$= (e_1, -\langle \alpha_1, \alpha_0^\vee \rangle f_1) = 2.$$

In total, one may check that

$$(t^i X, t^j Y) = \delta_{i+j,0}(X, Y) = \delta_{i+j,0}\, \mathrm{tr}(XY) \quad \text{for } X, Y \in \mathfrak{sl}_2.$$

and $(c, \mathcal{L}\mathfrak{sl}_2) = (d, \mathcal{L}\mathfrak{sl}_2) = 0$. $\square$

<u>Exercise:</u> Experiment with the formula $[x, y] = (x, y)\, \nu^{-1}(\alpha)$

for $x \in \mathfrak{g}_\alpha$, $y \in \mathfrak{g}_{-\alpha}$ for $\mathfrak{g} = \widehat{\mathcal{L}\mathfrak{sl}_2}$ $\square$

1.6. Constructing the Casimir Operator

Let $A = DB$ symmetrizable with quadruple $(\mathfrak{g}(A), \mathfrak{h}, \Pi, \Pi^\vee)$

and bilinear form $(\, , \,)$.

For each $\alpha \in \Delta$, we choose dual bases $\{e_\alpha^{(i)}\}$, $\{e_{-\alpha}^{(i)}\}$

of $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$, that is, $(e_\alpha^{(i)}, e_{-\alpha}^{(j)}) = \delta_{ij}$.

Moreover, fix $\rho \in \mathfrak{h}^*$, s.t. $\langle \rho, \alpha_i^\vee \rangle = \frac{1}{2} a_{ii}$. Then

$(\rho, \alpha_i) = \frac{1}{2}(\alpha_i, \alpha_i)$. Moreover let $\{u_i\}, \{u^i\}$ be

dual bases of $\mathfrak{h}$.

Lemma: Let $\alpha, \beta \in \Delta$. Then

(1) $(x, y) = \sum_i (x, e_{-\alpha}^{(i)})(y, e_\alpha^{(i)}) \quad \forall x \in \mathfrak{g}_\alpha, y \in \mathfrak{g}_{-\alpha}$.

(2) For all $z \in \mathfrak{g}_{\beta - \alpha}$

$$\sum_i e_{-\alpha}^{(i)} \otimes [z, e_\alpha^{(i)}] = \sum_i [e_{-\beta}^{(i)}, z] \otimes e_\beta^{(i)} \quad \text{in } \mathfrak{g}(A)^{\otimes 2} \text{ and } U(\mathfrak{g}(A))$$

$$\sum_i [e_{-\alpha}^{(i)}, [z, e_\alpha^{(i)}]] = \sum_i [[e_{-\beta}^{(i)}, z], e_\beta^{(i)}] \quad \text{in } \mathfrak{g}(A).$$

(3) $\sum_i \langle \lambda, u^i \rangle \langle \mu, u_i \rangle = (\lambda | \mu) \quad \forall \lambda, \mu \in \mathfrak{h}^*$

(4) $[\sum u^i u_i, x] = x\left((\alpha|\alpha) + 2\nu^{-1}(\alpha)\right) \quad \forall x \in \mathfrak{g}_\alpha$

Proof: Omitted $\square$

Lecture 10

Def: (1) Let V be a $\mathfrak{g}(A)$-module. Then V is called <u>restricted</u> if for all $v \in V$, $\mathfrak{g}_\alpha v = 0$ for almost all $\alpha \in \Delta^+$.

(2) For a restricted V-module. Let

$$\Omega_0 = 2 \sum_{\alpha \in \Delta_+} \sum_i e_{-\alpha}^{(i)} e_\alpha^{(i)} \in \mathrm{End}(V) \text{ and}$$

$$\Omega = 2\nu^{-1}(\rho) + \sum_i u^i u_i + \Omega_0 \in \mathrm{End}(V).$$

Then Ω is called the <u>generalized Casimir operator</u> on V. $\square$

Thm: Let V be a restricted $\mathfrak{g}(A)$-module.

(1) Let $u \in U(\mathfrak{g}'(A))_\alpha$, then

$$[\Omega_0, u] = -u\,(2(\rho,\alpha) + (\alpha,\alpha) + 2\nu^{-1}(\alpha)) \in \mathrm{End}(V)$$

(2) $\Omega \in \mathrm{End}_{\mathfrak{g}(A)}(V)$

Proof: (2) follows from (1) and Lemma (4)

(1) We show this by induction on $\mathrm{ht}(\alpha)$ for $\alpha \in Q$.

Assume that the formula holds for $a \in U(\mathfrak{g}'(A))_\alpha$ and $b \in U(\mathfrak{g}'(A))_\beta$. Then

$$[\Omega_0, ab] = [\Omega_0, a]b + a[\Omega_0, b]$$

$$[\Omega_0, ab] = [\Omega_0, a]b + a[\Omega_0, b]$$

$$= -a(2(\rho,\alpha) + (\alpha,\alpha) + 2v^{-1}(\alpha))b$$

$$- ab(2(\rho,\beta) + (\beta,\beta) + 2v^{-1}(\beta))$$

$$= -ab(2(\rho,\alpha) + (\alpha,\alpha) + 2v^{-1}(\alpha) + 2(\alpha,\beta)$$

$$+ 2(\rho,\beta) + (\beta,\beta) + 2v^{-1}(\beta))$$

$$= -ab(2(\rho,\alpha+\beta) + (\alpha+\beta,\alpha+\beta) + 2v^{-1}(\alpha+\beta)).$$

Since e_i, f_i generate $\mathfrak{g}'(A)$, it suffices to check the equation for $u = e_i, f_i$. In fact, we have by Lemma (2)

$$[\Omega_0, e_i] = 2 \sum_{\alpha \in \Delta_+} \sum \left([e^{(j)}_{-\alpha}, e_i] e^{(j)}_{\alpha} + e^{(j)}_{-\alpha} [e^{(j)}_{\alpha}, e_i] \right)$$

$$= 2 [f_i, e_i] e_i$$

$$\left. \begin{aligned} &+ 2 \sum_{\alpha \in \Delta_+ \setminus \{\alpha_i\}} \Big(\sum_i [e^{(j)}_{-\alpha}, e_i] e^{(j)}_{\alpha} + \\ &\qquad \sum_i e^{(j)}_{-\alpha+\alpha_i} [e^{(j)}_{\alpha-\alpha_i}, e_i] \Big) \end{aligned} \right\} \begin{array}{l} \text{Lemma (2)} \\ = 0 \end{array}$$

$$= 2 [f_i, e_i] e_i = -2 \nu^{-1}(\alpha_i) e_i$$

$$= \cdots = e_i \left(2(\rho, \alpha_i) + (\alpha_i, \alpha_i) + 2\nu^{-1}(\alpha_i) \right)$$

A similar proof applies to f_i. $\square$

<u>Cor.</u>: Let $v \in V$ be a highest weight vector. So $v \in V_\lambda$ for some $\lambda \in \mathfrak{h}^*$ and $e_i(v) = 0$ $\forall i$. Then

$$\Omega(v) = (\lambda + 2\rho, \lambda) v.$$

If v generates V, then

$$\Omega = (\lambda + 2\rho, \lambda) I_V \qquad \square$$

<u>Example:</u> Let $A = 0 \in \mathbb{C}^{n \times n}$. Then

$$\mathfrak{g} = \mathfrak{g}(0) = \underbrace{\sum_{i=1}^{n} \mathbb{C}\alpha_i^\vee}_{\mathfrak{h}'} \oplus \underbrace{\sum_{i=1}^{n} \mathbb{C} d_i}_{\mathfrak{h}''} \oplus \sum_{i=1}^{n} \mathbb{C} e_i \oplus \sum_{i=1}^{n} \mathbb{C} f_i$$

with Lie brackets

$$[e_i, e_j] = [f_i, f_j] = 0, \quad [e_i, f_j] = \delta_{ij} \alpha_i^\vee \quad \text{and}$$

$$[d_i, e_j] = \delta_{ij} e_j, \quad [d_i, f_j] = -\delta_{ij} f_j, \quad [\mathfrak{h}', \mathfrak{g}] = 0.$$

The bilinear form is given by

$$(e_i, f_i) = 1, \quad (\alpha_i^\vee, d_i) = 1 \quad \text{and} = 0 \text{ else}.$$

In this case, we can take $\rho = 0$ and

$$\Omega = 2\sum_i \alpha_i^\vee d_i + 2\sum_i f_i e_i .$$

Set $\mathfrak{h}_0' = \ker(\mathfrak{h}' \to \mathbb{C}, \alpha_i^\vee \mapsto 1)$. Then $\mathfrak{g}'(0)/\mathfrak{h}_0'$

is the Heisenberg Lie algebra (see 0.1 Example (6)) $\square$

1.7. $\mathfrak{sl}_2$-triples in Kac-Moody Algebras.

From now, we assume that $A=(a_{ij})$ is a generalized Cartan matrix with quadruple

$$(\mathfrak{g}(A), \mathfrak{h}, \Pi=\{\alpha_1,\dots,\alpha_n\}, \Pi^\vee=\{\alpha_1^\vee,\dots,\alpha_n^\vee\}).$$

Then, for each i, we obtain the $\mathfrak{sl}_2$-triple

$$\mathfrak{g}_{(i)} = \mathbb{C}e_i + \mathbb{C}\alpha_i^\vee + \mathbb{C}f_i \subset \mathfrak{g} .$$

From the representation theory of $\mathfrak{sl}_2$, we can deduce the following relations for $\mathfrak{g}(A)$.

<u>Thm:</u> For $i \neq j$ we have

$$(\operatorname{ad} e_i)^{1-a_{ij}} e_j = 0 \quad \text{and} \quad (\operatorname{ad} f_i)^{1-a_{ij}} f_j = 0 .$$

Proof: We show the second equation. Our goal is to show that for all $k = 1, \dots, n$ we get

$$(*)_k \qquad [e_k, (\mathrm{ad}\, f_i)^{1-a_{ij}} f_j] = 0$$

Then the relation follows from Lemma 1.4.

As in 0.9 we consider the $\mathfrak{g}_{(i)} \hat{=} \mathfrak{sl}_2$ - representation generated by f_j:

$$V = \langle (\mathrm{ad}\, f_i)^n f_j \mid n \rangle .$$

Since $[e_i, f_j] = 0$ and $[\alpha_i^\vee, f_j] = -a_{ij} \ni \mathbb{Z}_{\geq 0}$, we see that $M(-a_{ij}) \twoheadrightarrow V \twoheadrightarrow L(-a_{ij})$, where the lowest weight vector in $L(-a_{ij})$ corresponds to

$(\mathrm{ad}\, f_i)^{-a_{ij}} f_j$, and hence

$$[e_i, (\mathrm{ad}\, f_i)^{1-a_{ij}} f_j] = 0.$$

and $(*)_i$ holds. If $k \neq i, j$, $(*)_k$ is clear

from the relation $[e_k, f_i] = 0$ and $[e_k, f_j] = 0$.

For $k = j$, there are two cases.

Case $a_{ij} \neq 0$: Exercise!

Case $a_{ij} = 0$: Then $a_{ji} = 0$ and hence

$$[e_j, [f_i, f_j]] = a_{ji} f_i = 0 \qquad \square$$

Lecture 11

1.8 Integrability

We want to make sense of expressions like

$$\exp(x) = \sum_{i=0}^{\infty} \frac{x^i}{i!} \quad \text{for } x \in \mathfrak{g}.$$

<u>Def</u>: (1) Let V be a $\mathfrak{g}$-module and $x \in \mathfrak{g}$.

Then x is called <u>locally nilpotent</u> on V, if for all $v \in V$ there is an $N \geq 0$, s.t. $x^N v = 0$.

In this case, we define the operator

$$\exp(x) = I_V + x + \frac{x^2}{2} + \dots$$

on V.

(2) Let M be a $\mathfrak{g}(A)$-module. Then

M is called <u>integrable</u>, if M is a weight module (= diagonalizable for h) and e_i, f_i are locally nilpotent on V $\square$

<u>Lemma</u> (1) If $x_1, x_2, \dots$ generate g and for all $x \in g$, $(\operatorname{ad} x)^{n_i} x_i = 0$ for some n_i, then $\operatorname{ad} x$ is locally nilpotent on g.

(2) If $v_1, v_2, \dots \in V$ generate a g-module V and for $x \in g$, $x^{n_i} v = 0$ for some $n_i \geq 0$ <u>and</u> $\operatorname{ad} x$ is locally nilpotent on g, then x is loc. nilpotent on V.

(3) $\operatorname{ad} e_i$ and $\operatorname{ad} f_i$ are locally nilpotent on $\mathfrak{g}(A)$

<u>Proof</u>: (1) (2) Omitted

(3) Use Serre relations 1.7 Thm,

$(\operatorname{ad} e_i)^2 h = (\operatorname{ad} f_i)^2 h = 0$ for $h \in \mathfrak{h}$ and (1). $\square$

<u>Cor</u>: The adjoint rep. of $\mathfrak{g}(A)$ is integrable $\square$

1.9 Integrability and the Weyl group

<u>Def</u>: For $i=1,\dots,n$, the <u>fundamental/simple reflection</u> is the operator on $\mathfrak{h}^*$ defined by

$$r_i(\lambda) = \lambda - \langle \lambda, \alpha_i^\vee \rangle \alpha_i .$$

The <u>Weyl group</u> $W \subset GL(\mathfrak{h}^*)$ is the group generated by the r_i $\square$

<u>Thm</u>: Let V be an integrable rep. of $\mathfrak{g}(A)$. Then:

(1) As a $\mathfrak{g}_{(i)}$-module V decomposes as a direct sum of f.d. irred. $\mathfrak{h}$-invariant $\mathfrak{g}_{(i)}$-modules.

(2) The action of $\mathfrak{g}_{(i)} \cong \mathfrak{sl}_2$ can be integrated

to an action of $SL_2(\mathbb{C})$, that is, $SL_2(\mathbb{C})$ acts on V and the differential of the actions yields the $\mathfrak{g}_{(i)}$-action.

(3) The element $\begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \in SL_2(\mathbb{C})$ corresponds to

$$r_i^V = (\exp f_i)(\exp -e_i)(\exp f_i)$$

and for $\lambda \in \mathfrak{h}^*$

$$r_i^V(V_\lambda) = V_{r_i(\lambda)}$$

(4) The set of weights of V is W-invariant

(5) For $V = \mathfrak{g}(A)$, $r_i^{\mathfrak{g}} \in \mathrm{Aut}(\mathfrak{g}(A))$ and $r_i^{\mathfrak{g}}|_{\mathfrak{h}} = r_i$

(6) $(\ ,\ )_{\mathfrak{h}\times\mathfrak{h}}$ is W-invariant

Proof: (1) For $v \in V$, consider

$$U = \sum_{k,m \geq 0} \mathbb{C} f_i^k e_i^m (v)$$

Then U is $(\mathfrak{g}_{(i)} + \mathfrak{h})$-invariant. Since e_i and f_i act locally nilpotent on V, $\dim U < \infty$.

By 0.6. Thm, U decomposes into a direct sum of irreducibles.

(2) - (4) Follow from f.d. rep. th. of sl_2

(5) (6) Omitted $\square$

Remark: If V is integrable, we can consider the group $\mathcal{G}^V \subset GL(V)$ generated by the copies of $SL_2(\mathbb{C})$ for each $\mathfrak{g}_{(i)}$. If the kernel of the representation is contained in $\mathfrak{h}$, we can consider $\mathcal{G}^V$ as a first approximation of our Kac-Moody group.

For example, for $V = \mathfrak{g}$ we obtain the "adjoint" Kac-Moody group $\mathcal{G}^{ad}$. However $\mathfrak{z} = \ker ad$.

To fix this, consider the free group $\mathcal{G}^*$ generated by $x \in \mathfrak{g}(A)$. Let

$$\mathcal{G}(A) = \mathcal{G}^* / \bigcap_{\pi} \ker \pi$$

where π ranges over all integrable representations.

Then $\mathfrak{g}(A)$ is a central extension of $\mathfrak{g}^{ad}$

We will later return to this problem $\square$

1.10 Weyl groups - Tits Cone

<u>Def</u>: (1) $(r_{i_1}, \ldots, r_{i_t})$ is called called a <u>reduced expression</u> of $w \in W$, if $w = r_{i_1} r_{i_2} \cdots r_{i_t}$ and t is minimal amongst all expressions of w. Then $\ell(w) = t$ is the <u>length of w</u>.

(2) The <u>fundamental chamber</u> is the subset

$$C = \{ h \in \mathfrak{h}_{\mathbb{R}} \mid \langle \alpha_i, h \rangle \geq 0 \ \forall i \} \subset \mathfrak{h}_{\mathbb{R}}$$

For $w \in W$, $w(C)$ is called a chamber and

$$X = \bigcup_{w \in W} w(C)$$

is the <u>Tits cone</u>. Dually we get $C^\vee, X^\vee, \ldots$

Lemma: (1) $\ell(w r_i) < \ell(w)$ if and only if $w(\alpha_i) < 0$.

(2) (Exchange condition) Let $(r_{i_1}, \dots, r_{i_t})$ a reduced expression of $w \in W$. If $\ell(w r_i) < \ell(w)$, then there is an $1 \leq s \leq t$, s.t.

$$r_{i_s} r_{i_{s+1}} \cdots r_{i_t} = r_{s_{i+1}} \cdots r_{i_t} r_i$$

(3) For $h \in C$, $W_h = \{ w \in W \mid w(h) = h \} = \langle r_i \mid r_i(h) = h \rangle$

(4) C is a fundamental domain of X

(5) $X = \{ h \in \mathfrak{h}_{\mathbb{R}} \mid \langle \alpha, h \rangle \geq 0$ for almost all $\alpha \in \Delta_+ \}$

In particular X is a convex cone.

(6) $C = \{h \in \mathfrak{h}_{\mathbb{R}} \mid$ for all $w \in W$, $h - w(h) = \sum c_i \alpha_i^\vee$, where $c_i \geq 0\}$

(7) The following are equivalent

(a) $|W| < \infty$ (b) $X = \mathfrak{h}_{\mathbb{R}}$ (c) $|\Delta| < \infty$ (d) $|\Delta^\vee| < \infty$.

(8) If $h \in X$, then $|W_h| < \infty \iff h$ is in the interior of X.

<u>Proof</u> Omitted $\square$

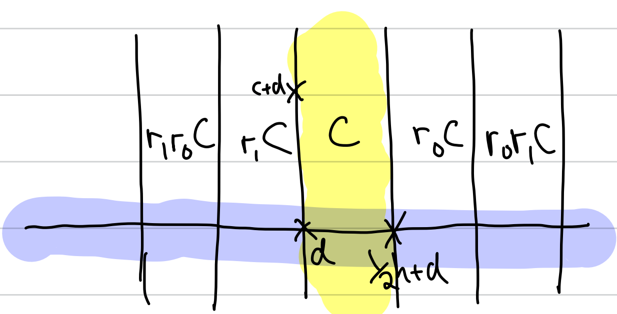
Example let $A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$, $\mathfrak{h}_{\mathbb{R}} = \mathbb{R}h \oplus \mathbb{R}c \oplus \mathbb{R}d$.

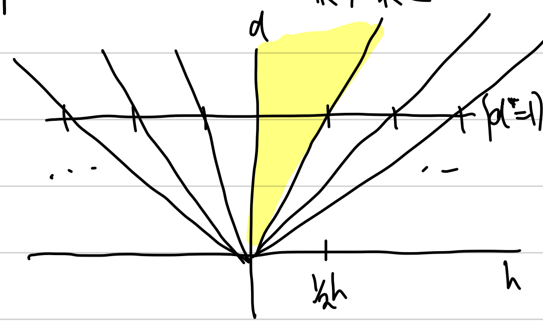
We have $C = \{ x_0 h + x_1 c + x_2 d \mid x_2 - 2x_0 \geq 0,\ x_0 \geq 0 \}$

The reflections r_0, r_1 are given by

$$r_0 = \begin{array}{c} \begin{array}{ccc} h & c & d \end{array} \\ \begin{pmatrix} -1 & & 1 \\ 1 & 1 & \\ & & 1 \end{pmatrix} \end{array} \qquad r_1 = \begin{pmatrix} -1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$$

We draw the $\{d^* = 1\}$ hyperplane and $\mathfrak{h}_{\mathbb{R}} / \mathbb{R}c$

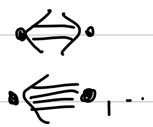




We see that $X = \{ d^* > 0 \} \cup \mathbb{R}c$ $\square$

Lecture 12

Prop: The Weyl group W is a Coxeter group with presentation $W = \langle r_i \mid r_i^2 = (r_i r_j)^{m_{ij}} = 1 \rangle$ where

$a_{ij}a_{ji}$	0	1	2	3	≥ 4
Dynkin					
Example	$\mathfrak{sl}_2 \times \mathfrak{sl}_2$	$\mathfrak{sl}_3$	$\mathfrak{sp}_4$	$\mathfrak{g}_2$	$\tilde{L}\mathfrak{sl}_2, A_2^{(2)}, \ldots$
m_{ij}	2	3	4	6	∞

Interlude: Overview of progress

$$\Delta_- \subset \underset{\alpha_i}{h^*} \supset \Delta_+$$

$$\Big|\ a_{ij} = \langle \alpha_i, \alpha_j^\vee \rangle \longleftarrow A \text{ symm.} \Rightarrow (\ ,\ ),\ r,\ \Omega$$

$$A = (a_{ij})$$

$$\Big\downarrow$$

$$\tilde{g}(A)/r = g(A) = \underbrace{\underset{f_i}{n_-} \oplus \underset{\alpha_i^\vee}{h} \oplus \underset{e_i}{n_+}}$$

$$\Big\uparrow$$

A GCM $\Rightarrow$ Serre:

$$(\operatorname{ad} e_i)^{1-a_{ij}} e_j = 0$$

$$(\operatorname{ad} f_i)^{1-a_{ij}} f_j = 0$$

$a_{ii} = 2 \Rightarrow g_{(i)} \cong sl_2$

$a_{ii} = 0 \Rightarrow$ Heisenberg

V integrable $\Rightarrow$

$V_\lambda \cong V_{w(\lambda)}$

$$\Big\uparrow$$

A GCM $\Rightarrow$ $$W = \langle \underset{g_{(i)}}{r_i} \mid r_i^2 = 1, (r_i r_j)^{\overset{a_{ij}a_{ji}}{m_{ij}}} = 1 \rangle$$

$$\Big\updownarrow$$

$$\bigcup w(C) = X \subset h_{\mathbb{R}}$$

1.11 Classification

<u>Def. + Thm.:</u> Let A be an indecomposable GCM.

Then A is either of <u>finite</u>, <u>affine</u> or <u>indefinite</u>

type where

(Fin) $\det(A) \neq 0$; there exists $u > 0$ such that $Au > 0$;

$$Av \geq 0 \Rightarrow v > 0 \text{ or } v = 0$$

(Aff) $\operatorname{corank} A = 1$; there exists $u > 0$ such that $Au = 0$;

$$Av \geq 0 \Rightarrow Av = 0;$$

(Ind) there exists $u > 0$ such that $Au < 0$; $Av \geq 0, v \geq 0$

$$\Rightarrow v = 0$$

<u>Proof</u>: Omitted $\square$

<u>Lemma</u>: Let A be an indecomposable GCM. Then

(1) A is symmetrizable

(2) A is affine if and only if

(a) All proper principal minors are $\neq 0$ and $\det A = 0$

(b) A is pos. semidefinite of corank $= 1$

(c) There is a unique $\delta \in \mathbb{Z}^n_{\geq 0} \setminus \{0\}$, s.t.,

$$\ker A \cap \mathbb{Z}^n = \mathbb{Z}\delta$$

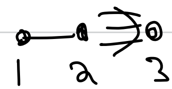
<u>Proof</u>: Omitted $\square$

Thm: The Dynkin diagrams of affine GCM's are

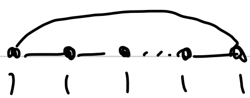
Table Aff 1

$A_1^{(1)}$

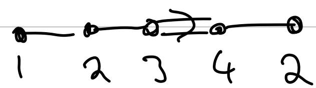
$G_2^{(1)}$



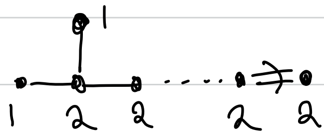
$A_\ell^{(1)}$



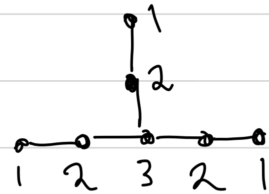
$F_4^{(1)}$



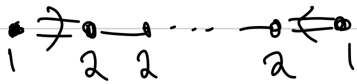
$B_\ell^{(1)}$



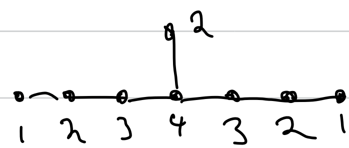
$E_6^{(1)}$



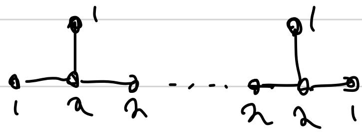
$C_\ell^{(1)}$



$E_7^{(1)}$



$D_\ell^{(1)}$



$E_8^{(1)}$

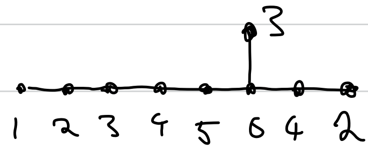
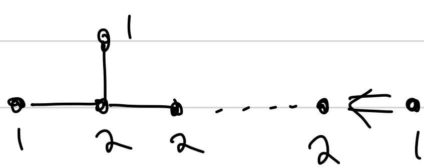


Table Aff2

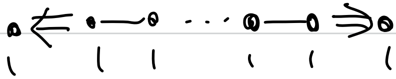
$A_2^{(2)}$

$A_{2\ell}^{(2)}$

$A_{2\ell-1}^{(2)}$



$D_{\ell+1}^{(2)}$



$E_6^{(2)}$

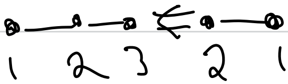
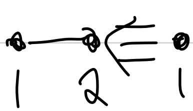


Table Aff3

The numbers under the diagram are the coordinates of δ.

$D_4^{(3)}$



<u>Rem</u>: The affine Dynkin diagrams in Table Aff 1 arise from extending the corresponding finite Dynkin diagram by a root $\alpha_0 \hat{=} -\theta$. In the finite root system

$$\theta = \sum_{i=1}^{\ell} a_i \alpha_i$$

is the highest root, that is, $\theta + \alpha_i \notin \Delta \ \forall i$. $\square$

1.12 Real and Imaginary Roots

Again, let A be GCM with $(\mathfrak{g}(A), \mathfrak{h}, \Pi, \Pi^\vee)$.

<u>Def</u>: The set of roots is partitioned in <u>real roots</u>

$$\Delta^{re} = W\Pi$$

and imaginary roots

$$\Delta^{im} = \Delta \setminus \Delta^{re}.$$

If $\alpha = w(\alpha_i) \in \Delta^{re}$, we denote by $\alpha^\vee = w(\alpha_i^\vee) \in \Delta^{re,\vee}$ the <u>dual root</u> and by

$$r_\alpha = w r_i w^{-1}; \ \mathfrak{h}^* \to \mathfrak{h}^*, \ \lambda \mapsto \lambda - \langle \lambda, \alpha^\vee \rangle \alpha$$

the corresponding reflections. $\square$

Remark: For each $\alpha = w(\alpha_i) \in \Delta^{re}$ with $w = r_{i_1} \cdots r_{i_t}$

we get a copy of sl_2 via

$$\mathfrak{g}_{(\alpha)} = r_{i_1}^{\mathfrak{g}(A)} \cdots\cdots r_{i_t}^{\mathfrak{g}(A)}\left(\mathfrak{g}_{(i)}\right)$$

Prop: (1) $\dim \mathfrak{g}_\alpha = 1 \quad \forall \alpha \in \Delta^{re}$

(2) $\mathbb{Z}\alpha \cap \Delta = \{\pm\alpha\} \quad \forall \alpha \in \Delta^{re}$

(3) For $\alpha \in \Delta^{re}$ and $\beta \in \Delta$, $(\beta + \mathbb{Z}\alpha) \cap (\Delta \cup \{0\})$

is an α-root string through β of the form

$$\beta + p\alpha, \dots, \beta + q\alpha$$

with $p + q = \langle \beta, \alpha^\vee \rangle$.

(4) If $\pm\alpha \in \Delta^{re} \setminus \Pi$, then there is an r_i, s.t.

$$|\operatorname{ht} r_i\alpha| < |\operatorname{ht} \alpha|$$

(5) Δ_+^{im} is W-invariant

(6) For $\alpha \in \Delta^+_{im}$, then there is a unique $\beta \in -C^\vee$

(so $\langle \beta, \alpha_i^\vee \rangle \leq 0 \ \forall i$), s.t., $\alpha = w(\beta)$ for $w \in W$.

(7) Let A be symmetrizable, then

(a) $(\alpha, \alpha) > 0$ and $\alpha^\vee = 2\frac{\nu^{-1}(\alpha)}{(\alpha,\alpha)}$ for all $\alpha \in \Delta^{re}$

(b) $(\alpha, \alpha) \leq 0 \iff \alpha \in \Delta^{im}$ for all $\alpha \in \Delta$.

Proof: (1)–(3) and (7)(a) are clear for simple roots. Now conjugate with w.

(4) Assume the contrary and wlog. $\alpha > 0$. Then $-\alpha \in C^\vee \subset \mathfrak{h}^\vee$. Let $w \in W$, s.t. $w(\alpha) \in \Pi$. Then

$-\alpha + w(\alpha) = \sum c_i \alpha_i > 0$ with $c_i \geq 0$ by 1.10 lemma (6)

But now also $-\alpha + w(\alpha) < 0$, since $w(\alpha) \in \Pi$. ↯

(5) Since $\Delta_+ \setminus \{\alpha_i\}$ is r_i-invariant,

$\Delta_+^{im} = \Delta_+ \setminus (W(\Pi) \cap \Delta_+)$ is W invariant.

(6) Let $\alpha \in \Delta_+^{im}$. Let β be of minimal height in $W\alpha \subset \Delta_+$. Then $-\beta \in C^\vee$. Since $C^\vee$ is a fundamental domain, such β is unique.

(7) (b) "$\Leftarrow$" follows from (7)(a)

"$\Rightarrow$" Let $\alpha \in \Delta_+^{im}$. Wlog. $-\alpha \in C^\vee$.

Then $\alpha = \sum_i h_i \alpha_i \quad h_i \geq 0$ and

$$(\alpha,\alpha) = \sum_i h_i (\alpha, \alpha_i) = \sum_i \underbrace{\tfrac{1}{2}|\alpha_i|^2}_{>0} \underbrace{h_i}_{\geq 0} \underbrace{\langle \alpha, \alpha_i^\vee \rangle}_{\leq 0} \leq 0$$

□

Lecture 13

Thm A: For $\alpha = \sum k_i \alpha_i \in Q$, let $\operatorname{supp}(\alpha)$ be the subdiagram of the Dynkin diagram spanned by the i with $k_i \neq 0$. Let

$$K = \{ \alpha \in Q_+ \setminus \{0\} \mid \langle \alpha, \alpha_i^\vee \rangle \leq 0 \ \forall i \text{ and } \operatorname{supp} \alpha \text{ is connected} \}$$

Then:

(1) $K \subset \Delta_+^{im}$ (2) $\Delta_+^{im} = \bigcup_{w \in W} w(K)$

(3) Let $\alpha \in \Delta_+^{im}$, then $\mathbb{Q}\alpha \cap Q \subset \Delta^{im} \cup \{0\}$

Proof: (1) Omitted

(2) "$\supseteq$" (a) and Prop. (5). "$\subseteq$" Prop. (6)

(3) from (1) $\square$

Thm B: Let A be an indecomposable GCM

(1) Let A be of finite type. Then

(a) $\Delta^{im} = \emptyset$ (b) $X = \mathfrak{h}_{\mathbb{R}}$

(1) Let A be of affine type. Then

(a) $\Delta^{im} = \mathbb{Z}\delta \setminus \{0\}$ (b) $X = \{h \in \mathfrak{h}_{\mathbb{R}} \mid \langle \delta, h \rangle > 0\}$

Here $\delta = \sum_{i=1}^{n} a_i \alpha_i$, where $a_i > 0$, $\gcd\{a_i\} = 1$, $A(a_i) = 0$.

Proof: Omitted $\square$

Example: $A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$, $\mathfrak{h} = \mathbb{C}h \oplus \mathbb{C}c \oplus \mathbb{C}d$, then

$\Delta_{im} = \mathbb{Z}\delta$, where $\delta = d^*$ $\square$

1.13 Non-twisted Affine Lie Algebras

We explain how to construct the non-twisted affine Kac-Moody algebras $X_\ell^{(1)}$ from Table Aff 1, see 1.11 Thm.

<u>Step 1</u> For $P = \sum c_k t^k \in \mathcal{L} = \mathbb{C}[t, t^{-1}]$, denote the residuum by

$$\mathrm{Res}(P) = c_{-1}$$

and for $P, Q \in \mathcal{L}$, the bilinear form

$$\varphi(P, Q) = \mathrm{Res}\left(\frac{dP}{dt} Q\right)$$

so that $\varphi(t^n, t^m) = \delta_{n+m,0}\, n$ and for $P, Q, R \in \mathcal{L}$

$$\varphi(P, Q) = -\varphi(Q, P), \quad \varphi(PQ, R) + \varphi(QR, P) + \varphi(RP, Q) = 0.$$

<u>Step 2</u>: Let A be the Cartan matrix associated to a Dynkin diagram $X_\ell^{(1)}$ $(X = A, \dots, G)$ in Table Aff 1.

Here, we label the rows/colums $0, \dots, \ell$, s.t. removing the 0th row/column yields the Cartan matrix $\mathring{A}$ of type X_ℓ. Denote by $\mathring{\mathfrak{g}} = \mathfrak{g}(\mathring{A})$ the corresponding simple f.d. Lie algebra. □

<u>Step 3</u>: Denote by the <u>loop algebra</u> of $\mathring{\mathfrak{g}}$ by

$$\mathcal{L}\mathring{\mathfrak{g}} = \mathbb{C}[t, t^{-1}] \otimes \mathring{\mathfrak{g}}$$

with Lie bracket

$$[P \otimes x, Q \otimes y] = PQ\,[x, y]$$

□

<u>Step 4</u>: Let $(,)$ be a non-deg. $\mathring{g}$-invariant bilinear form (so a non-zero multiple of the Killing form). This extends to a $\mathbb{C}[t,t^{-1}]$-bilinear form on $\mathcal{L}\mathring{g}$ by

$$(P\otimes x, Q\otimes y) = PQ\,(x,y)$$

Moreover, the derivation $d_s = t^s \frac{d}{dt}$ extends to $\mathcal{L}\mathring{g}$ via

$$d_s(P\otimes x) = d_s(P)\otimes x. \qquad \square$$

<u>Step 5</u>: We define a 2-cocycle on $\mathcal{L}\mathring{g}$ via

$$\psi(P\otimes x, Q\otimes y) = (x,y)\,\varphi(P,Q).$$

Then, indeed ψ is a cocycle, that is,

$$\psi(a,b) = -\psi(a,b) \text{ and } \psi([a,b],c) + \psi([b,c],a) + \psi([c,a],b) = 0$$

This follows directly from the properties of φ and $(,)$. $\square$

<u>Step 6</u>: 2-cocycles of a Lie algebra yield central extensions.

We define $\mathcal{L}\mathring{g} = \mathcal{L}\mathring{g} \oplus \mathbb{C}c$ the central extension corresponding to ψ, so that for $a,b \in \mathcal{L}\mathring{g}$ and $\lambda, \mu \in \mathbb{C}$

$$[a+\lambda c, b+\mu c] = [a,b] + \psi(a,b)c. \qquad \square$$

<u>Step 7</u>: Any derivation of a Lie algebra can be added to it.

Let $d = d_0 = t\frac{d}{dt}$. Then we define the affine Lie algebra by

$$\hat{\mathcal{L}}\mathring{g} = \mathcal{L}g \oplus \mathbb{C}c \oplus \mathbb{C}d$$

with Lie bracket for $x,y \in g$, $\alpha, \beta, \lambda, \mu \in \mathbb{C}$

$[t^m \otimes x + \lambda c \oplus \alpha d, t^n \otimes y + \mu c + \beta d] =$

$t^{m+n}[x,y] + \alpha n t^n \otimes y - \beta m t^m \otimes x + m\,\delta_{m-n,0}\,(x,y)\,c$ $\square$

We now show that $\hat{\mathcal{L}}\mathring{\mathfrak{g}} = \mathfrak{g}(A)$. Denote by

$\{\alpha_1, \dots, \alpha_\ell\} = \mathring{\Delta} \subset \mathring{\mathfrak{h}}^*$ the roots of $\mathring{\mathfrak{g}}$, by E_i, F_i the

Chevalley generators and $H_i = [E_i, F_i]$ the coroots.

Denote by $\theta = \sum_{i=1}^{\ell} a_i \alpha_i \in \Delta$ the highest root of $\mathring{\mathfrak{g}}$

(see Remark 1.11). Let $F_0 \in \mathring{\mathfrak{g}}_\theta$, s.t., $(F_0, \mathring{\omega}(F_0)) = \frac{-2}{(\theta,\theta)}$.

and $E_0 = -\mathring{\omega}(F_0)$, so that $[E_0, F_0] = -\theta^\vee$.

Now $E_0, \dots, E_\ell$ generate $\mathring{\mathfrak{g}}$, since the adjoint rep. is

a lowest weight representation with lowest weight $E_0 \in \mathfrak{g}_{-\theta}$.

Denote by $\mathfrak{h} = 1 \otimes \mathring{\mathfrak{h}} \oplus \mathbb{C}c \oplus \mathbb{C}d$ and let

$\delta \in \mathfrak{h}^*$ with $\delta(\mathring{\mathfrak{h}} + \mathbb{C}c) = 0$ and $\delta(d) = 1$. Moreover let

$$e_0 = t \otimes E_0,\ f_0 = t^{-1} \otimes F_0,\quad e_i = 1 \otimes E_i,\ f_i = 1 \otimes F_i .$$

Then $[e_0, f_0] = \frac{2}{(\theta,\theta)} c - \theta^\vee$ Also, $\hat{\mathfrak{L}}\mathring{\mathfrak{g}}$ has a roots

$$\Delta = \{ j\delta + \gamma,\ j \in \mathbb{Z},\ \gamma \in \mathring{\Delta} \} \cup \{ j\delta,\ j \in \mathbb{Z} \setminus 0 \}$$

where $(\mathfrak{L}\mathring{\mathfrak{g}})_{j\delta+\gamma} = t^j \otimes \mathring{\mathfrak{g}}_\gamma$ and $(\mathfrak{L}\mathring{\mathfrak{g}})_{j\delta} = t^j \otimes \mathfrak{h}$.

Now, set

$$\Pi = \{ \alpha_0 = \delta - \theta, \alpha_1, \dots, \alpha_\ell \} \text{ and}$$

$$\Pi^\vee = \{ \alpha_0^\vee = \frac{2}{(\theta,\theta)} c - 1 \otimes \theta^\vee, \alpha_1^\vee = 1 \otimes H_1, \dots, \alpha_\ell^\vee = 1 \otimes H_\ell \}.$$

Then indeed, $A = (\langle \alpha_j, \alpha_i^\vee \rangle)_{i,j}$.

Thm: The algebra $\hat{\mathcal{L}}\mathring{g}$ is isomorphic to $g(A)$, with Cartan subalgebra, Π, $\Pi^\vee$ and e_i, f_i defined as above.

Proof: It is easy to check that the $e_i, f_i, \alpha_i^\vee$ fulfill the relations defining $\tilde{g}(A)$, so we obtain a map

$$\tilde{\pi}: \tilde{g}(A) \to \hat{\mathcal{L}}\mathring{g}.$$

Assume that $0 \neq i \subset \hat{\mathcal{L}}\mathring{g}$ is an ideal with $h \cap i = 0$.

Then there is an $\alpha \in \Delta$, s.t. $i \cap \hat{L}(\mathring{g})_\alpha \neq 0$. So

$t^j \otimes x \in i$ for some $j \in \mathbb{Z}$, $0 \neq x \in \mathring{g}_\gamma$ and $\gamma \in \mathring{\Delta} \cup \{0\}$.

Choose $y \in g_{-\gamma}$, s.t. $(x, y) \neq 0$. Then

$$[t^j \otimes x, t^{-j} \otimes y] = j(x,y)c + [x,y] \in h \cap i = \{0\}.$$

Hence $j=0$, so that $\gamma \neq 0$. But then $[x,y] \neq 0$ $\lightning$

This shows, that $\hat{\pi}$ descends to a map

$$\pi : g(A) \to \hat{\mathcal{L}} \mathring{g}$$

which induces an isomorphism on the Cartan h

To see that π is an isomorphism, it suffices to show that e_i, f_i, $i = 0, \dots, l$ and h generate $\hat{\mathcal{L}} \mathring{g}$.

This is a nice exercise! $\square$

When normalizing $(\,,\,)$ by $(\theta,\theta)=2$ we obtain

$$(P\otimes x, Q\otimes y) = \operatorname{Res}(t^{-1}PQ)\,(x,y)$$

as well as

$$(\mathbb{C}c\oplus\mathbb{C}d, \mathcal{L}\mathring{g})=0 \quad \text{and } (c,c)=(d,d)=0,\ (c,d)=1 .$$

Moreover, $\hat{\mathcal{L}}g$ has the triangular decomposition

$$\hat{\mathcal{L}}g = t^{-1}\mathbb{C}[t^{-1}]\otimes(\mathring{n}_+ + \mathring{h}) + \mathbb{C}[t^{-1}]\otimes\mathring{n}_- \quad \leftarrow n^-$$

$$\oplus\, h \,\oplus\, t\mathbb{C}[t]\otimes(\mathring{n}_- + \mathring{h}) + \mathbb{C}[t]\otimes\mathring{n}_+ \in n^+$$

the Chevalley involution is

$$\omega(P(t)\otimes x + \lambda c + \alpha d) = P(t^{-1})\otimes\mathring{\omega}(x) - \lambda c - \alpha d$$

Lecture 14

1.14 Twisted affine Lie algebras.

We sketch an explicit construction of the twisted affine Lie algebras $X_N^{(k)}$, $k=2,3$, see Thm. 1.11.

Denote the Cartan matrix by A.

<u>Step 1</u>: Let X_N be the underlying finite Dynkin diagram, so

$A_{2\ell}, A_{2\ell-1}, D_{\ell+1}, E_6$ $(k=2)$ or D_4 $(k=3)$.

Then X_N has an automorphism σ of order k, that yields an automorphism σ of the corresponding simple f.d. Lie algebra $\mathring{\mathfrak{g}}$. $\square$

<u>Step 2</u>: Denote by $\mathring{\mathfrak{g}}_{\bar{j}}$ the $e^{2\pi i j/k}$-eigenspace. This

yields a $\mathbb{Z}/k$-grading of $\mathring{g}$, so

$$\mathring{g} = \bigoplus_{\bar{j} \in \mathbb{Z}/k} g_{\bar{j}} .$$

The pairing $(\,,\,)$ pairs $g_{\bar{i}}$ and $g_{\bar{j}}$ for $\bar{i}+\bar{j}=\bar{0}$ $\square$

Step 3: We denote

$$\mathcal{L}(\mathring{g},\sigma) = \bigoplus_{j \in \mathbb{Z}} t^j \otimes g_{\bar{j}} \subset \mathcal{L}\mathring{g} .$$

This is the fixed point algebra of $\mathcal{L}\mathring{g}$ with respect to the automorphism $\tilde{\sigma}(t^j \otimes x) = e^{-2\pi i j/k}\, t^j \otimes \sigma(x)$. $\square$

Step 4 Recall the algebra $\hat{\mathcal{L}}\mathring{g} = \mathcal{L}g \oplus \mathbb{C}c_0 \oplus \mathbb{C}d_0$ (we rename c and d). Then let

$$\hat{\mathcal{L}}(\mathring{g},\sigma) = \mathcal{L}(\mathring{g},\sigma) \oplus \mathbb{C}c_0 \oplus \mathbb{C}d_0$$ $\square$

To see that $\hat{\mathcal{L}}(\mathfrak{g},\sigma) = \mathfrak{g}(X_N^{(k)})$ one

constructs Chevalley generators and the Cartan:

The fixed points $\mathring{\mathfrak{g}}_{\bar{0}} = (\mathring{\mathfrak{g}})^{\sigma}$ are also a simple

Lie algebra obtained by "folding".

Let $\mathfrak{h}_0 \subset \mathring{\mathfrak{g}}_{\bar{0}}$ the Cartan. Moreover $\mathring{\mathfrak{g}}_{\bar{0}}$ has ℓ

Chevalley generators E_i, F_i. To obtain the additional generator,

choose an (appropriatly scaled) lowest weight

vector E_ε in the $\mathring{\mathfrak{g}}_{\bar{0}}$-rep. $\mathring{\mathfrak{g}}_{T}$, and let

$F_\varepsilon = \mathring{\omega}(E_\varepsilon)$.

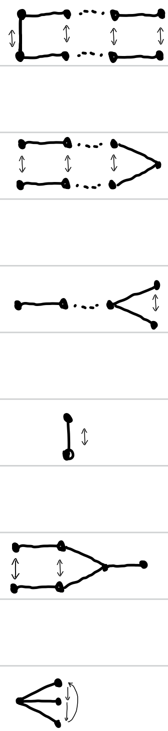
Then we obtain a Cartan $\mathfrak{h} = \mathfrak{h}_0 \oplus \mathbb{C}c_0 \oplus \mathbb{C}d_0$

as well as generators $e_i = 1 \otimes E_i$, $f_i = 1 \otimes F_i$,

$e_\varepsilon = t \otimes E_\varepsilon$, $f_\varepsilon = t^{-1} \otimes F_\varepsilon$.

<u>Sketch:</u>

$\mathring{\mathfrak{g}}$	σ	$\mathring{\mathfrak{g}}_{\bar 0}$	$\hat{\mathcal{L}}(\mathring{\mathfrak{g}}, \sigma)$	
$A_{2\ell}$		B_ℓ		$A_{2\ell}^{(2)}$
$A_{2\ell-1}$		C_ℓ		$A_{2\ell-1}^{(2)}$
$D_{\ell+1}$		B_ℓ		$D_{\ell+1}^{(2)}$
A_2		A_1		$A_2^{(2)}$
E_6		F_4		$E_6^{(2)}$
D_4		G_2		$D_4^{(3)}$



$/\!/\!/ = 1 \otimes E_i \quad \mathring{\mathfrak{g}}_{\bar 0}$

$/\!/\!/ = t \otimes E_\varepsilon \quad \mathring{\mathfrak{g}}_{\bar 1}$

Now, $\hat{\mathcal{L}}(g, \sigma)$ has roots given by

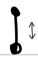
$$\Delta = \{ j\delta + \gamma,\ j \in \mathbb{Z},\ \gamma \in \Delta(\mathring{g}_{\bar{j}}) \} \cup \{ j\delta,\ j \in \mathbb{Z} \setminus \{0\} \}$$

where $\hat{\mathcal{L}}(\mathring{g}, \sigma)_{j\delta+\gamma} = t^j \otimes g_{\bar{j}, \gamma}$, $\hat{\mathcal{L}}(\mathring{g}, \sigma)_{j\delta} = t^j \otimes g_{\bar{j}, 0}$.

<u>Thm</u>: The algebra $\hat{\mathcal{L}}(\mathring{g}, \sigma)$ is isomorphic to $g(A)$, with Cartan subalgebra, Π, $\Pi^\vee$ and e_i, f_i defined as above.

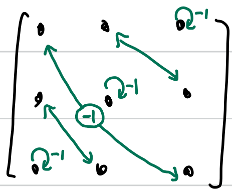
<u>Proof</u> Omitted $\square$

<u>Cor</u>: Let $g(A)$ be an affine Lie algebra of rank $l+1$ of type $X_N^{(k)}$. Then $\dim g_{jk\delta} = l$ for $k \neq 0$ and $\dim g_{(jk+s)\delta} = \frac{(N-l)}{(k-1)}$ for $k \neq 1$, $s \neq 0$ $\square$

Example: $A_2^{(2)} \hat{=} \begin{bmatrix} 2 & -4 \\ -1 & 2 \end{bmatrix}$  $\rightsquigarrow$ 

Let $\mathring{\mathfrak{g}} = \mathfrak{sl}_3$.

Then σ acts on the entries of a 3×3 matrix in this way:



Hence, the invariants and anti-invariants are

$$\mathfrak{sl}_2 \cong \mathring{\mathfrak{g}}_{\bar 0} = \left\{ \begin{bmatrix} a & b & \\ c & & b \\ & c & -a \end{bmatrix} \right\} \xrightarrow[\text{on}]{\text{acts}} L(4) \cong \mathring{\mathfrak{g}}_{\bar 1} = \left\{ \begin{bmatrix} & y & z \\ w & x & -y \\ v & -w & \end{bmatrix} \right\}$$

and we obtain (next to $\mathfrak{h} = \mathfrak{h}_0 \oplus \mathbb{C}c_0 \oplus \mathbb{C}d_0$) generators of $\hat{\mathcal{L}}(\mathfrak{sl}_3, \sigma)$

$$e_0 = \begin{bmatrix} & & \\ & & \\ t & & \end{bmatrix} \quad \text{and} \quad e_1 = \begin{bmatrix} & 1 & \\ & & 1 \\ & & \end{bmatrix}$$

Hence, $\langle \alpha_1, \alpha_0^\vee \rangle = -\dim L(4) + 1 = -4$. Exercise: Experiment more!

Lecture 15

1.15 - The affine Weyl group

Let A be Cartan matrix of affine type as in Thm. 1.11.

Then there a unique vectors (a_i), $(a_i^\vee)$, with $a_i^{(\vee)} > 0$

and $\gcd(a_i) = \gcd(a_i^\vee)$ with

$$A(a_i) = A^{tr}(a_i^\vee) = 0$$

called the <u>label</u> and <u>colabel</u> of A.

Then A can be symmetrized via $D = \operatorname{diag}(\varepsilon_i)$

with $\varepsilon_i := \frac{a_i}{a_i^\vee}$. This yields a symmetric bilinear

form on $\mathfrak{h}$ where:

$$a_i\, \nu(\alpha_i^\vee) = a_i^\vee\, \alpha_i\,, \quad (\alpha_i, \alpha_j) = \frac{a_i^\vee}{a_i}\, a_{ij}\,, \quad (\alpha_i^\vee, \alpha_j^\vee) = \frac{a_j}{a_j^\vee}\, a_{ij}.$$

We denote the canonical central element and fundamental imaginary root by

$$c = \sum_{i=0}^{\ell} a_i^\vee \alpha_i^\vee \quad \text{and} \quad \delta = \sum_{i=0}^{\ell} a_i \alpha_i .$$

For simplicity let A be of type $X_N^{(1)}$, and $\mathring{A}$ the matrix of type X_N, obtained by removing the 0th row/column. Then c and δ coincide with the elements constructed in 1.13 and

$$\theta = \sum_{i=1}^{\ell} a_i \alpha_i = \delta - \alpha_0 , \quad \theta^\vee = \sum_{i=1}^{\ell} a_i^\vee \alpha_i^\vee = c - \alpha_0^\vee$$

are the highest root and the associated coroot of the Lie algebra $\mathring{g} = g(\mathring{A})$ of type X_N.

Let $\mathring{W} = \langle r_i \mid i=1,\dots,n \rangle$ the corresponding finite Weyl group and $\mathring{Q}^\vee = \bigoplus_{i=1}^{\ell} \mathbb{Z}\alpha_i^\vee$ the coroot lattice.

Then the Weyl group of $\mathfrak{g}(A) = \hat{\mathcal{L}}(\mathring{\mathfrak{g}})$ is

$$W = \langle \mathring{W}, r_0 \rangle.$$

Denote by $r_\theta \in \mathring{W}$ the reflection associated to the root θ. Let $t_{\theta^\vee} = r_0 r_\theta \in W$, so that

$$W = \langle \mathring{W}, t_{\theta^\vee} \rangle.$$

For $h \in \mathring{\mathfrak{h}}$, define $T_h \in \mathrm{End}(\mathfrak{h})$ via

$$T_h(\lambda) = \lambda + \langle \lambda, c \rangle \nu(h) - \left(\langle \lambda, h \rangle + \tfrac{1}{2}(h,h)\langle \lambda, c \rangle \right)\delta.$$

Then $T_h T_{h'} = T_{h+h'}$ and $w T_h w^{-1} = T_{wh}$ for $w \in \mathring{W}$.

Modulo $\mathbb{C}\delta$ and on the hyperplane $\{\langle -, c\rangle = 1\}$

T_h is simply a translation by $\nu(h)$.

Moreover, one can show that

$$t_{\theta^\vee} = T_{\theta^\vee}.$$

Since $\mathbb{Z}\mathring{W}\theta^\vee = \mathring{Q}^\vee$, we obtain an isomorphism

$$\mathring{Q}^\vee \xrightarrow{\sim} \langle\langle t_{\theta^\vee} \rangle\rangle_W \qquad \theta^\vee \mapsto t_{\theta^\vee}$$

of the coroot lattice and the normal closure of $t_{\theta^\vee}$ in W. Hence, we get an isomorphism

$$\mathring{W} \ltimes \mathring{Q}^\vee \longrightarrow W.$$

This group is also called the <u>affine Weyl group</u>.

For the twisted case $X_N^{(2,3)}$, the Weyl group admits a similar description as

$$W \cong \mathring{W}_{\bar{0}} \ltimes \mathring{Q}_{\bar{0}}$$

where $\mathring{W}_{\bar{0}}$ is the Weyl group of the "folded" Dynkin diagram associated to the Lie algebra $\mathring{\mathfrak{g}}_{\bar{0}}$, and $\mathring{Q}_{\bar{0}}$ is the root lattice.

1.16 Category $\mathcal{O}$ - Basics

Let A be any $n \times n$-matrix. Recall that

$\mathfrak{g}(A) = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$.

<u>Def A</u>: (1) For a weight module (= diagonizable over $\mathfrak{h}$)

$V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda$, denote the set of weights by

$$P(V) = \{\lambda \in \mathfrak{h}^* \mid V_\lambda \neq 0\}$$

(2) For $\lambda \in \mathfrak{h}^*$, let

$$D(\lambda) = \{\mu \in \mathfrak{h}^* \mid \mu \leq \lambda\} = \lambda - \sum_i \mathbb{Z}_{\geq 0} \alpha_i = \lambda - Q_+$$

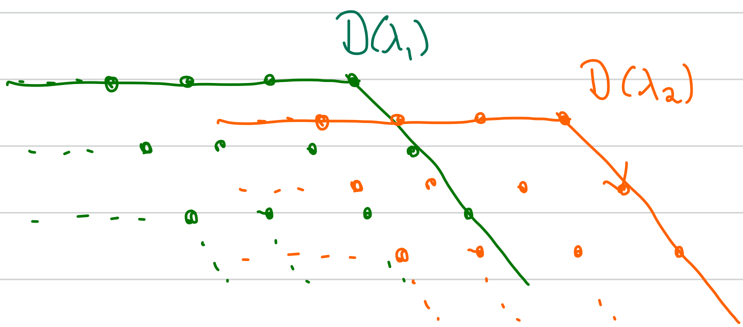
(3) <u>Category $\mathcal{O}$</u> is the full subcategory of $\mathfrak{g}(A)$-modules of all modules V, s.t.,

(a) V is a weight module with $\dim V_\lambda < \infty \ \forall \lambda \in \mathfrak{h}$

(b) There are $\lambda_1, \dots, \lambda_s \in \mathfrak{h}^*$, s.t.,

$$P(V) \subseteq \bigcup_{i=1}^{s} D(\lambda_i)$$

<u>Sketch:</u>



Def B. Let M be a $\mathfrak{g}(A)$-module. For $\lambda \in \mathfrak{h}^*$,

$0 \neq v^+ \in V_\lambda$ is called a highest weight vector

if $\mathfrak{n}^+ v^+ = 0$. If M is generated by v^+,

M is called a highest weight module

(with highest weight λ).

Proposition: (1) Category $\mathcal{O}$ is closed with respect to:

(a) submodules (b) quotients (c) direct sums (d) tensor products

(2) Every module in $\mathcal{O}$ is restricted

(3) Every highest weight module is in $\mathcal{O}$

Lecture 16

<u>Proof</u> (1) For a short exact sequence

$0 \to V_1 \to V_2 \to V_3 \to 0$ of $\mathfrak{g}$-modules,

V_2 diagonizable for $\mathfrak{h}$ implies V_1, V_3 are, too.

Moreover $P(V_2) = P(V_1) \cup P(V_2)$.

This shows (a) - (c).

Let V_1, V_2 be $\mathfrak{h}$-diagonalizable. Then

$$V_1 \otimes V_2 = \bigoplus_{\lambda \in \mathfrak{h}^*} \bigoplus_{\substack{\mu, \mu' \\ \mu + \mu' = \lambda}} (V_1)_\mu \otimes (V_2)_{\mu'}$$

is also $\mathfrak{h}$-diagonalizable.

Let $P(V_1) \subset \bigcup_i D(\lambda_i)$, $P(V_2) \subset \bigcup_j D(\lambda'_j)$.

Then $P(V_1 \otimes V_2) \subset \bigcup_{i,j} D(\lambda_i + \lambda'_j)$.

This shows (d)

(2) Let $V \in \mathcal{O}$ and $v \in V$. Wlog. $v \in V_\lambda$ for some $\lambda \in \mathfrak{h}^*$. Assume that $P(V) = \bigcup_i D(\lambda_i)$.

Let $\alpha \in Q_+$. If $\lambda + \alpha \notin \bigcup_i [\lambda, \lambda_i] = S$

$\lambda + \alpha \notin P(V)$, so $\mathfrak{g}_\alpha v \subset V_{\lambda+\alpha} = 0$.

Since S is finite, $\mathfrak{g}_\alpha v = 0$ for almost all α.

(3) Let M be a highest weight module with highest weight vector v^+ of weight λ.

Then by PBW, $M = U(n^-)v^+$ and

$$P(M) \subset D(\lambda)$$

since $P(U(n^-)) = D(0) = -Q_+$ $\square$

<u>Lemma:</u> ("Schur's Lemma") Let M be a highest weight module of highest weight λ. Then restriction

$$\mathrm{End}_{\mathfrak{g}}(V) \to \mathrm{End}_{\mathbb{C}}(V_\lambda) = \mathbb{C}$$

yields an isomorphism

<u>Proof.</u> Clear $\square$

1.17 Verma and simple modules - Definition

<u>Def A:</u> let $\lambda \in \mathfrak{h}^*$. Denote by $\mathbb{C}_\Lambda$ the 1-dim. rep. of $\mathfrak{h}+\mathfrak{n}_+$, with $(h+n)(v) = \Lambda(h)v$. Then the <u>Verma module</u> of <u>highest weight</u> Λ is

$$M(\Lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{h}+\mathfrak{n}_+)} \mathbb{C}_\Lambda$$
$$= U(\mathfrak{g}) / J(\Lambda)$$

where $J(\Lambda) = U(\mathfrak{g}) \{ (h - \Lambda(h)) + n \mid h \in \mathfrak{h}, n \in \mathfrak{n}_+ \}$ $\square$

<u>Prop:</u> let $\Lambda \in \mathfrak{h}^*$.

(1) $M(\Lambda)$ is the unique (up to iso.) module, s.t.

for each highest weight module V with

highest weight λ, there is a surjective map

$$M(\Lambda) \to V$$

(2) As a $\mathcal{U}(n^-)$-module $M(\Lambda) \cong \mathcal{U}(n^-)$

(3) $P(M(\Lambda)) = D(\Lambda)$ and $M(\Lambda) \in \mathcal{O}$

(4) $M(\Lambda)$ contains a unique proper submodule $M'(\Lambda)$.

<u>Proof</u>: (1) Clear. (2) PBW-theorem (0.2 Thm)

(3) Since $\mathcal{U}(n^-)$ is generated by $g_{-\alpha_i}$ for $i=1,\dots,n$

$P(\mathcal{U}(n^-)) = -Q_+$. By (2), $P(M(\Lambda)) = \Lambda - Q_+$

(4) Let $M \subsetneq M(\Lambda)$ be a proper submodule. Then by 1.4 Prop. $M = \bigoplus M_\lambda$. Since $M(\Lambda)$ is generated by $M(\Lambda)_\lambda = \mathbb{C}\lambda$, $M_\lambda = 0$. This shows that the sum of proper submodules is proper. Hence we can take $M'(\Lambda) = \sum_{M \subsetneq M(\Lambda)} M$ $\square$

<u>Def B</u>: We denote the simple highest weight module of highest weight $\Lambda \in \mathfrak{h}^*$ by

$$L(\Lambda) = M(\Lambda)/M'(\Lambda)$$ $\square$

1.18 Primitive Vectors and simple modules

<u>Def</u>: Let V be a $\mathfrak{g}(A)$-module. Then $v \in V_\lambda$ is called <u>primitive</u> of weight λ if there is a submodule $U \subset V$, s.t., $v \notin U$ and $\mathfrak{n}_+ v \in U$.

Then λ is called a primitive weight. $\square$

<u>Prop</u>: Let $0 \neq V \in \mathcal{O}$. Then

(1) V contains a highest weight ($\Rightarrow$ primitive) vector.

(2) The following are equivalent

(a) V is irred.

(b) V is a highest weight module and all

$v \in V$ primitive are highest weight vectors generating V.

(c) $V \cong L(\Lambda)$ for some $\Lambda \in \mathfrak{h}^*$

(3) V is generated by primitive vectors.

<u>Proof</u>: (1) Take $0 \neq v \in V_\lambda$ for $\lambda \in P(V)$ maximal

(2) (a) $\Rightarrow$ (b) Let V be irreducible. Let $0 \neq v \in V_\lambda$ be a highest weight vector (see (1)). Then $U(\mathfrak{g})v = U(\mathfrak{n}_-)v = V$, so V is a h.w. module and $V_\lambda = 1$. If $w \in V_\mu$ is primitive, w is a h.w. vector. If $\mu \neq \lambda$, $\mu < \lambda$ and

$0 \neq U(\mathfrak{g})\, w = U(\mathfrak{n}^-) \subsetneq V$ ↯ Hence $\mu = \lambda$ and $\mathbb{C}v = \mathbb{C}w = V_\lambda$

(b) $\Rightarrow$ (c) Assume that V fulfills (ii), so V is a highest weight module for some weight $\lambda \in \mathfrak{h}^*$ and we obtain a surjection

$$q : M(\lambda) \to V.$$

Now let $0 \neq U \subset V$ be a submodule. By (i) U contains a primitive vector. By assumption this vector generates V, so $U = V$. Hence V is irred. and q induces an isomorphism $L(\lambda) \xrightarrow{\sim} V$

(c) $\Rightarrow$ (a) Clear.

(3) Denote by $V' \subset V$ the submodule generated by all primitive vectors. Assume that $V' \neq V$.

Let $0 \neq v \in V_\lambda \setminus V'_\lambda$ with λ maximal. Then $\mathfrak{n}^+ v \subseteq V'$ by construction. So v is primitive $\lightning$ $v \notin V'$ $\square$

<u>Cor.</u> The map $\lambda \mapsto L(\lambda)$ yields a bijection

$$\mathfrak{h}^* \xrightarrow{1:1} \{\text{irred. in } \mathcal{O}\}/_{\text{iso}} \qquad \square$$

1.19 Contravariant form

For a weight module $V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda$, the naive dual

$V^* \cong \prod_{\lambda \in \mathfrak{h}} (V_\lambda)^*$ is not necessarily a weight module

anymore. Moreover $(\mathbb{C}_\lambda)^* = \mathbb{C}_{-\lambda}$, so this duality

does not preserve $\mathcal{O}$. Instead, we consider

$$V^\vee = \bigoplus_{\lambda \in \mathfrak{h}} (V_\lambda)^* = \{ f \in V^\vee \mid f(V_\lambda) = 0 \text{ f.a.a. } \lambda \in \mathfrak{h}^* \}$$

where we twist the $\mathfrak{g}(A)$-action

$$x \cdot_\omega f = \omega(x) \cdot f = f(-\omega(x) \cdot -)$$

via the Cartan involution. Recall that $\omega(h) = -h$

for $h \in \mathfrak{h}$. Hence $(\mathbb{C}_\lambda)^\vee = \mathbb{C}_\lambda$ and $P(M^\vee) = P(M)$

so that $(\ )^\vee$ defines a duality on $\mathcal{O}$. Now

$L(\Lambda)^\vee$ is also irreducible of highest weight Λ,

so there is (up to scalar) a unique isomorphism

$$L(\Lambda) \to L(\Lambda)^\vee.$$

Since $L(\Lambda)^\vee \subset L(\Lambda)^*$, this defines a pairing B on $L(\Lambda)$,

called the <u>contravariant form</u>, such that

$$B(x\cdot v, w) = -B(v, \omega(x)\cdot w) \quad \forall x \in \mathfrak{g},\ v, w \in L(\Lambda).$$

The form is symmetric, non-deg. and descends to

a non-deg. form on each weight space.

<u>Exercise</u>: Give a formula for B in terms of $B(v^+, v^+)$, for $v^+ \in L(\Lambda)_\Lambda$

Lecture 17

1.20 Filtrations and Multiplicities

Lemma: Let $V \in \mathcal{O}$.

(1) If for any two primitive weights λ, μ we have

$\lambda \geq \mu$ implies $\lambda = \mu$, then V is completely reducible.

(2) Let $\lambda \in \mathfrak{h}^*$. Then, there is a filtration

$$V = V_t \supset V_{t-1} \supset \dots \supset V_1 \supset V_0 = 0$$

and a subset $J \subset \{1, \dots, t\}$, such that

(a) if $j \in J$, then $V_j / V_{j-1} \cong L(\lambda_j)$ for some $\lambda_j \geq \lambda$

(b) if $j \notin J$, then $(V_j / V_{j-1})_\mu = 0$ for $\mu \geq \lambda$.

Proof (1) Let $V^\circ = V^{\mathfrak{n}^+} = \{v \in V \mid \mathfrak{n}^+ v = 0\}$. This is $\mathfrak{h}$-invariant and has a decomposition $V^\circ = \bigoplus_{\lambda \in \mathcal{L}} V^\circ_\lambda$, where all

$\lambda \in \mathcal{L}$ are primitive weights.

Let $0 \neq v \in V^\circ_\lambda$. Let $\bar{V} = U(g) v = U(n^-) v$.

Assume that $0 \neq W \subset \bar{V}$. Then by 1.18 Prop (1),

W contains a primitive vector $0 \neq w \in V^\circ_\mu \cap W$. Then

$\mu \leq \lambda$, so $\mu = \lambda$ and $W = \bar{V}$.

Hence $\bar{V} = U(g) V^\circ$ is completely reducible.

Assume that $V/\bar{V} \neq 0$. Let μ be maximal with

$V_\mu / \bar{V}_\mu \neq 0$. By maximality $n_+ v \subset \bar{V}$, so v and μ

are primitive. Since $v \notin V^\circ$, there is an i, s.t. $e_i v \in \bar{V}$.

So the $\mu + \alpha_i \leq \lambda$ for some $\lambda \in \mathfrak{h}$.

Hence $\lambda \geq \mu + \alpha_i > \mu$ $\lightning$ $\lambda = \mu$.

(2) Exercise: Induction on $a(V, \lambda) = \sum_{\mu \geq \lambda} \dim V_\mu$. $\square$

<u>Def:</u> Let $V \in \mathcal{O}$ and $\mu \in \mathfrak{h}^*$. Let $\lambda \leq \mu$ and choose a filtration as in Lemma (2). Then we define the multiplicity of $L(\mu)$ in V as

$$[V : L(\mu)] = \#\{j \mid \lambda_j = \mu\}. \qquad \square$$

So $[V : L(\mu)] \neq 0 \iff \mu$ is a primitive weight.

1.21 Formal characters

<u>Def</u>: (1) Denote by $e(\lambda) \in \mathrm{Fun}(\mathfrak{h}^*, \mathbb{C})$ the indicator function of λ. Let $\Sigma \subset \mathrm{Fun}(\mathfrak{h}^*, \mathbb{C})$ the subspace of all functions that vanish outside some $\bigcup_{i=1}^{s} D(\lambda_i)$. Then Σ is an algebra w.r.t. convolution

$$\sum_\lambda c_\lambda e(\lambda) \cdot \sum_\lambda c'_\lambda e(\lambda) = \sum_\lambda \Big(\sum_{\mu+\nu=\lambda} c_\mu c'_\nu\Big) e(\lambda).$$

(2) Let $V \in \mathcal{O}$. Then the <u>formal character</u> of V is

$$\mathrm{ch}\, V = \sum_{\lambda \in \mathfrak{h}^*} (\dim V_\lambda)\, e(\lambda) \in \Sigma$$ $\square$

<u>Proposition</u>: (1) If $0 \to V' \to V \to V'' \to 0$ is s.e.s. in $\mathcal{O}$

$\mathrm{ch}\, V = \mathrm{ch}\, V' + \mathrm{ch}\, V''$

(2) For $V \in \mathcal{O}$

$$\operatorname{ch} V = \sum [V : L(\lambda)] \operatorname{ch} L(\lambda)$$

(3) For $\Lambda \in \mathfrak{h}^*$, we have

$$\operatorname{ch} M(\lambda) = e(\lambda) \prod_{\alpha \in \Delta_+} (1 + e(-\alpha) + e(-2\alpha) + \cdots)^{\operatorname{mult} \alpha}$$

$$= e(\lambda) \prod_{\alpha \in \Delta^+} (1 - e(-\alpha))^{-\operatorname{mult} \alpha}$$

Proof (1) Clear. (2) Use (1) and 1.20 lemma (2)

(3) Use that as $U(\mathfrak{n}^-)$-module $M(\Lambda) \cong U(\mathfrak{n}^-)$, see

1.17 Prop (2), and PBW for $U(\mathfrak{n}^-)$. $\square$

Remark: We obtain an isomorphism of rings

$$K_0(\mathcal{O}) \otimes \mathbb{C} \to \Sigma, \quad [M] \mapsto ch(M)$$

where

$$K_0(\mathcal{O}) = \mathbb{Z}\,\mathrm{Ob}(\mathcal{O}) / \langle [V'] + [V''] = [V], \text{ for } 0 \to V' \to V \to V'' \to 0 \rangle$$

is the Grothendieck group of $\mathcal{O}$.

1.22 Integrable highest weight modules

Now assume that A is a GCM.

<u>Definition</u>: The <u>weight lattice</u> and the <u>(regular) dominant integral weights</u> are

$$P = \{\lambda \in \mathfrak{h}^* \mid \langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z} \ \forall i\}$$

$$P_+ = \{\lambda \in P \mid \langle \lambda, \alpha_i^\vee \rangle \geq 0 \ \forall i\}$$

$$P_{++} = \{\lambda \in P \mid \langle \lambda, \alpha_i^\vee \rangle > 0 \ \forall i\}$$

□

<u>Recall</u>: For $\mathfrak{g} = \mathfrak{sl}_2$ and $\Lambda \in \mathfrak{h}^* = \mathbb{C}$,

$L(\Lambda)$ f.d. $\Leftrightarrow$ $L(\Lambda)$ integrable $\Leftrightarrow$ $M(\Lambda)$ reducible

$\Leftrightarrow \Lambda \in \mathbb{Z}_{\geq 0}$

<u>Lemma:</u> Let L be an integrable h.w. module of highest weight Λ. Then L integrable $\Leftrightarrow$ $\Lambda \in P_+$

<u>Proof:</u> Sketch: For $0 \neq v \in L_\lambda$ a highest weight vector, $U(\mathfrak{g}_{(i)})v$ is a highest weight module of $\mathfrak{sl}_2 \cong \mathfrak{g}_{(i)}$ of highest weight $\langle \lambda, \alpha_i^\vee \rangle$.

Now use the facts from Recall $\square$

<u>Corollary</u>: Let L be an highest weight module of highest weight $\Lambda \in P_+$. Then

(1) $\dim L_\mu = \dim L_{w\mu}$

(2) $w \operatorname{ch} L = \operatorname{ch} L$

(3) $w P(L) = P(L)$

(4) For $\mu \in P(L)$, there is a $x \in W$, s.t.

$$x\mu \in P(L) \cap P_+$$

$\square$

Lecture 18

1.23 Weyl-Kac character formula

Now, let A be a symmetrizable GCM. Denote by

$(\,,\,)$ the associated bilinear form and Ω the Casimir.

<u>Recall</u> - Cor. 1.6: for a highest weight vector $0 \neq v \in V_\Lambda$

in a restricted $\mathfrak{g}$-module V we have.

$$\Omega(v) = (\Lambda + 2\rho, \Lambda)\, v$$

If v generates V, $\Omega = \mathrm{Id}_V (\Lambda + 2\rho, \Lambda)$

Note that $(\Lambda + 2\rho, \Lambda) = |\Lambda + \rho|^2 - |\rho|^2$.

- $\rho \in \mathfrak{h}^*$ fulfills $(\alpha_i, \rho) = \frac{1}{2}(\alpha_i, \alpha_i)$ $\forall i$

- $r_i(\rho) = \rho - \alpha_i$ $\square$

We now prove some essential lemmata.

<u>Lemma A</u>: Let $R = \prod_{\alpha \in \Delta_+} (1 - e(-\alpha))^{\text{mult}\,\alpha} = ch(M(0))$

(this is the <u>Weyl denominator</u>). Let $w \in W$. Then

$$w(e(\rho)R) = \epsilon(w)\, e(\rho)\, R$$

where $\epsilon(w) = (-1)^{\ell(w)}$.

<u>Proof</u>: It suffices to treat the case $w = r_i$. Then

$$s_i(e(\rho)R) = e(\rho - \alpha_i) \prod_{\alpha \in \Delta_+} (1 - e(s_i(-\alpha)))^{\text{mult}\,\alpha}$$

$$= e(\rho) e(-\alpha_i)(1 - e(\alpha_i))\, R\, (1 - e(-\alpha_i))^{-1}$$

$$= -e(\rho) R = \epsilon(r_i) R \qquad \square$$

<u>Lemma B</u>: Let V be a highest weight module of highest weight Λ. Then

$$\operatorname{ch} V = \sum_{\substack{\lambda \leq \Lambda \\ |\lambda+\rho|^2 = |\Lambda+\rho|^2}} c_\lambda \operatorname{ch} M(\lambda)$$

for $c_\lambda \in \mathbb{Z}$, $c_\Lambda = 1$

<u>Proof</u>: By 1.21 Prop (2) it suffices to treat $V = L(\Lambda)$.

Let $B(\Lambda) = \{\lambda \leq \Lambda \mid |\lambda+\rho|^2 = |\Lambda+\rho|^2\}$. Order the set as $\lambda_1, \lambda_2, \dots$, s.t. $\lambda_i \geq \lambda_j$ implies $i \leq j$.

Now by 1.21 Prop (2) and 1.6 Cor.

$$\operatorname{ch} M(\lambda_i) = \sum_{\lambda_j \in B(\Lambda)} c_{ij} L(\lambda_j).$$

The matrix (c_{ij}) is triangular with $c_{ii}=1$.

Inverting the matrix yields the statement $\square$

<u>Lemma C</u> Let $\Lambda, \lambda \in P$, such that $\lambda \leq \Lambda$ and $\Lambda + \lambda \in P_+$.

Then either

(a) $\langle \Lambda + \lambda, \alpha_i^\vee \rangle = 0 \ \forall i \in \mathrm{supp}(\Lambda - \lambda)$ or

(b) $|\Lambda|^2 - |\lambda|^2 > 0$.

So, if $\Lambda \in P_{++}$, $\lambda \in P_+$ and $|\Lambda|^2 = |\lambda|^2$, then $\Lambda = \lambda$.

<u>Proof</u>: Write $\lambda = \Lambda - \beta$ for $\beta = \sum k_i \alpha_i$, $k_i \geq 0$.

Then we obtain

$$|\Lambda|^2 - |\lambda|^2 = \sum_{i \in \mathrm{supp}(\Lambda - \lambda)} \frac{1}{2} \underbrace{k_i}_{>0} \underbrace{(\alpha_i, \alpha_i)}_{>0} \underbrace{\langle \Lambda + \lambda, \alpha_i^\vee \rangle}_{\geq 0} .$$

If this is ≤ 0, $\langle \Lambda + \lambda, \alpha_i^\vee \rangle = 0 \ \forall i \in \mathrm{supp}(\Lambda - \lambda)$ $\square$

Thm: (Weyl-Kac character formula). Let L be an integrable highest weight module of highest weight Λ. Then, we get

$$\operatorname{ch} L = \operatorname{ch} L(\Lambda) = \frac{\sum_{w \in W} \epsilon(w)\, e(w \cdot \lambda)}{\prod_{\alpha \in \Delta_+} (1 - e(-\alpha))^{\operatorname{mult} \alpha}}$$

where $w \cdot \lambda = w(\lambda + \rho) - \rho$. Also, $L \cong L(\Lambda)$.

Proof: As in Lemma A, let R be the Weyl denominator. By Lemma B,

$$e(\rho) R \operatorname{ch}(L) = \sum_{\substack{\lambda \le \Lambda \\ |\lambda+\rho|^2 = |\Lambda+\rho|^2}} c_\lambda\, e(\lambda + \rho).$$

where $c_\Lambda = 1$, $c_\lambda \in \mathbb{Z}$. Now, by lemma A and 1.22 Cor. (2) the LHS is W- skew invariant.

Hence the same holds for the RHS, so that

$$c_\lambda = \epsilon(w) c_\mu \quad \text{if} \quad w(\lambda+\rho)-\rho = w\cdot\lambda = \mu$$

Let $\lambda \in \mathfrak{h}^*$, s.t., $c_\lambda \neq 0$. Then $c_{w\cdot\lambda} \neq 0$ and hence

$w\cdot\lambda \leq \Lambda$ and $|w\cdot\lambda+\rho| = |w(\lambda+\rho)| = |\Lambda+\rho|$ $\forall\, w \in W$.

Let $\mu \in W\cdot\lambda$ with $ht(\Lambda-\mu)$ minimal.

Hence, $ht(\Lambda-\mu) \geq ht(\Lambda - r_i\cdot\mu)$.

This implies that $\langle \mu+\rho, \alpha_i^\vee\rangle \geq 0$ (Exercise).

Hence $\mu+\rho \in P_+$. Since $\Lambda \in P_+$, $\Lambda+\rho \in P_{++}$.

Since $|\mu+\rho| = |w\cdot\lambda+\rho| = |\Lambda+\rho|$, we can apply lemma C and obtain $\mu=\Lambda$.

Hence $c_\lambda \neq 0 \Rightarrow \lambda \in W\cdot\Lambda$. Moreover, $c_\Lambda = 1$

So, we obtain

$$e(\rho) R \operatorname{ch} L = \sum_{w\in W} \epsilon(w)\, e(w(\Lambda+\rho)).$$

The remaining statement is clear $\square$

<u>Cor A:</u> By applying the Thm. to $L(0) = \operatorname{triv}$, we get the denominator identity

$$\prod_{\alpha\in\Delta_+} (1-e(-\alpha))^{\operatorname{mult}\alpha} = \sum_{w\in W} \epsilon(w)\, e(w\cdot 0)$$

Plugging this into the formula for $L(\Lambda)$ yields

$$\operatorname{ch} L(\Lambda) = \frac{\sum_{w \in W} \epsilon(w)\, e(w \cdot \Lambda)}{\sum_{w \in W} \epsilon(w)\, e(w \cdot 0)} \qquad \square$$

Expanding

$$\prod_{\alpha \in \Delta_+} (1 - e(\alpha))^{-\operatorname{mult} \alpha} = \sum_{\beta \in \mathfrak{h}^*} K(\beta)\, e(\beta)$$

yields the <u>Kostant partition function</u>.

In particular, $K(\beta)$ is the number of partitions of β into positive roots, where each root is counted with multiplicity. Also $K(\beta) = \dim M(0)_{-\beta}$.

Cor B (Kostant's formula): Let $\Lambda \in P_+$. Then

$$\dim L(\Lambda)_\lambda = \sum_{w \in W} \epsilon(w)\, K(w \cdot \Lambda - \lambda)$$

Proof: By the Weyl-Kac character formular

$$\mathrm{ch}\, L(\Lambda) = \Big(\sum_{w \in W} \epsilon(w)\, e(w \cdot \Lambda)\Big)\Big(\sum_{\beta \in h^*} K(\beta)\, e(-\beta)\Big)$$

$$= \sum_{w \in W} \sum_{\beta \in h^*} \epsilon(w) K(\beta) e(w \cdot \Lambda - \beta)$$

$$= \sum_{w \in W} \sum_{\lambda \in h^*} \epsilon(w)\, K(w \cdot \Lambda - \lambda)\, e(\lambda)$$

Collecting all coeff. of $e(\lambda)$ yields the statement $\square$

Lecture 19

1.24 Constructing functions from characters.

Fix a functional $s: Q \to \mathbb{Z}$. We indentify s with the tuple

$$s = (s_1, .., s_n) = (s(\alpha_1), \dots, s(\alpha_n))$$

Recall the $\mathbb{Z}$-grading

$$\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}_j(s) \quad , \text{where} \quad \mathfrak{g}_j(s) = \bigoplus_{\substack{\alpha \\ s(\alpha)=j}} \mathfrak{g}_\alpha .$$

Fix $\lambda_s \in \mathfrak{h}^*$ and $h^s \in \mathfrak{h}$, s.t.

$$\langle \lambda_s, \alpha_i^\vee \rangle = s_i \qquad \text{and} \quad \langle \alpha_i, h^s \rangle = s_i .$$

In particular if $s = \mathbb{1} = (1, \dots 1)$, $\lambda_s = \rho$ and $h^s = \rho^\vee$. In this case $\mathfrak{g}_j(\mathbb{1})$ is the principal grading.

Given s, we define the specialization map

$$F_s : \underset{\text{Laurent series}}{\mathbb{C}[\![e(-\alpha_1), \ldots, e(-\alpha_n)]\!]} \to \mathbb{C}[\![q]\!]$$

$$\underset{\mathcal{E}}{\cup}$$

by $F_s(e(-\alpha_i)) = q^{s_i}$, so that

$$F_s(e(\alpha)) = q^{\langle \alpha, h_s \rangle}.$$

1.25 Dual Lie algebra and dimension formula

If $(\mathfrak{h}, \Pi, \Pi^\vee)$ is a realization of A, then

$(\mathfrak{h}^*, \Pi^\vee, \Pi)$ —— " —— A^{tr}.

We call ${}^t\mathfrak{g} = \mathfrak{g}(A^{tr})$ the dual of $\mathfrak{g} = \mathfrak{g}(A)$.

In particular, $\mathfrak{h}^* \subset {}^t\mathfrak{g}$ is the Cartan,

$\Pi^\vee$ the roots, Π the coroots, ... of ${}^t\mathfrak{g}$.

Again let A be a symmetrizable GCM.

Prop: For the principal gradation $\mathbb{1} = (1, \dots, 1)$

$$\dim \mathfrak{g}_j(\mathbb{1}) = \dim {}^t\mathfrak{g}_j(\mathbb{1}).$$

Proof: Since $\dim \mathfrak{g}_j = \dim \mathfrak{g}_{-j}$ and

and $\mathfrak{g}_0(\mathbb{1}) = \mathfrak{h}$, ${}^t\mathfrak{g}_0(\mathbb{1}) = \mathfrak{h}^*$, we can assume $j \geq 1$.

We apply $F_{\mathbb{1}}$ to the denominator identity, 1.23 Cor A to obtain

$$\prod_{j \geq 1} (1 - q^j)^{\dim \mathfrak{g}_j(\mathbb{1})} = \sum_{w \in W(A)} \epsilon(w)\, q^{\langle w \cdot \rho, \rho^\vee \rangle}$$

$$\prod_{j \geq 1} (1 - q^j)^{\dim {}^t\mathfrak{g}_j(\mathbb{1})} = \sum_{w \in W({}^tA)} \epsilon(w)\, q^{\langle w \cdot \rho^\vee, \rho \rangle}$$

The RHS are equal, since $W(A) = W({}^tA)$ and

$$\begin{aligned}\langle w \cdot \rho, \rho^\vee \rangle &= \langle w(\rho), \rho^\vee \rangle - \langle \rho, \rho^\vee \rangle \\ &= \langle w(\rho^\vee), \rho \rangle - \langle \rho^\vee, \rho \rangle \\ &= \langle w \cdot \rho^\vee, \rho \rangle .\end{aligned}$$

Now solve the LHS for $\dim \mathfrak{g}_j(1)$. $\square$

We also see that

$$\prod_{\alpha\in\Delta_+} (1-q^{\langle\alpha,\rho^\vee\rangle})^{\mathrm{mult}\,\alpha} = \prod_{\alpha\in\Delta_+^\vee} (1-q^{\langle\rho,\alpha\rangle})^{\mathrm{mult}\,\alpha}.$$

<u>Prop B</u>: Let $\Lambda\in P_+$ and $s=(\langle\Lambda,\alpha_1^\vee\rangle,\dots,\langle\Lambda,\alpha_n^\vee\rangle)$. Then

$$F_{\mathbb{1}}(e(-\Lambda)\,\mathrm{ch}\,L(\Lambda)) = \prod_{\alpha\in\Delta_+^\vee} \frac{(1-q^{\langle\Lambda+\rho,\alpha\rangle})}{(1-q^{\langle\rho,\alpha\rangle})^{\mathrm{mult}(\alpha)}}$$

<u>Proof</u>: For $\lambda\in P_+$, set

$$N_\lambda = \sum_{w\in W} \epsilon(w)\, e(w(\lambda)-\lambda).$$

Then we get

$$F_{\mathbb{1}}(N_\lambda) = \sum_{w\in W} \epsilon(w)\, q_t^{\langle \lambda, \rho^\vee\rangle - \langle w(\lambda), \rho^\vee\rangle}$$

$$= \sum_{w\in W} \epsilon(w)\, q_t^{\langle \lambda, \rho^\vee - w(\rho^\vee)\rangle}$$

$$= F_r\Big(\sum_{w\in W} \epsilon(w)\, e(w\cdot\rho^\vee)\Big)$$

Cor A.1.23

$$= F_r\Big(\prod_{\alpha\in\Delta^\vee_+} (1-e(-\alpha))^{\operatorname{mult}\alpha}\Big)$$

$$= \prod_{\alpha\in\Delta^\vee_+} (1-q_t^{\langle\lambda,\alpha\rangle})^{\operatorname{mult}(\alpha)}$$

where $r = (\langle\lambda,\alpha_1^\vee\rangle, \ldots, \langle\lambda,\alpha_n^\vee\rangle)$. Applying this to the $e(-\Lambda)ch(\Lambda) = N_{\Lambda+\rho}/N_\rho$ yields the statement. $\square$

<u>Thm</u> (Weyl's dimension formula) Let $\mathfrak{g}$ be a semisimple Lie algebra, $\Lambda \in P_+$. Then

$$\dim L(\Lambda) = \prod_{\alpha \in \Delta_+^{\vee}} \frac{\langle \Lambda+\rho, \alpha \rangle}{\langle \rho, \alpha \rangle}$$

<u>Proof</u> Compute $\lim_{q \to 1}$ in Prop B $\square$

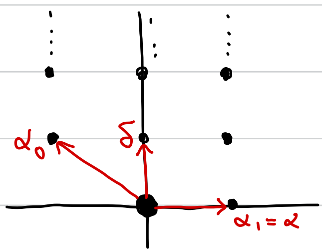
1.26 Jacobi's Triple Product Identity

We now study the Weyl denominator identity in the special case $A_1^{(1)} \cong \hat{\mathcal{L}}sl_2$.

Recall that $\Pi = \{\alpha_0, \alpha_1\}$ and

$$\Delta_+ = \{\alpha_0 + n\delta, \alpha_1 + n\delta, n \geq 0\} \cup \{n\delta, n \geq 0\}$$

$$= \{n\alpha_0 + (n-1)\alpha_1, (n-1)\alpha_0 + n\alpha_0, n(\alpha_0+\alpha_1), n \geq 1\}$$



Set $u = e(-\alpha_0)$ and $v = e(-\alpha_1)$, then

$$\prod_{\alpha \in \Delta^+} (1 - e(-\alpha)) = \prod_{n=1}^{\infty} (1 - u^n v^n)(1 - u^{n-1} v^n)(1 - u^n v^{n-1})$$

Moreover $W \cong \mathring{W} \ltimes M = \langle r_1 \rangle \ltimes \mathbb{Z}\alpha_1$ and

$$t_{n\alpha_1} \cdot \rho = 2n\alpha_1 - (2n^2 + n)\delta$$ and

$$t_{n\alpha_1} r_1 \cdot \rho = (2n-1)\alpha_1 - (2n^2 + n)\delta.$$

Thus, we get

$$\sum_{w \in W} \epsilon(w) e(w \cdot 0) = \sum_{n \in \mathbb{Z}} e(2n\alpha_1) e(-(2n^2 + n)\delta)$$

$$- \sum_{n \in \mathbb{Z}} e((2n-1)\alpha_1) e(-(2n^2 - n)\delta)$$

$$= \sum_{n \in \mathbb{Z}} (-1)^n u^{\frac{1}{2}n(n+1)} v^{\frac{1}{2}n(n-1)}.$$

The Weyl(-Kac) denominator identity yields

$$\prod_{n=1}^{\infty}(1-u^n v^n)(1-u^{n-1}v^n)(1-u^n v^{n-1})$$

$$= \sum_{n\in\mathbb{Z}} (-1)^n u^{\frac{1}{2}n(n+1)} v^{\frac{1}{2}n(n-1)}$$

Substituting $u = zq$ and $v = z^{-1}q$ yields the

Jacobi triple product identity

$$\prod_{n=1}^{\infty}(1-q^{2n})(1-z^{-1}q^{2n-1})(1-zq^{2n-1}) = \sum (-1)^n z^n q^{n^2}$$

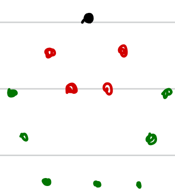
By putting $u = q^2$, $v = q$, we obtain the following expression of the Euler function

$$\varphi(q) = \prod_{n=1}^{\infty}(1-q^n) = \sum_{n\in\mathbb{Z}} (-1)^n q^{\frac{3n^2+n}{2}}$$

we can replace n by $-n$ in the RHS and obtain

$$\sum_{n\in\mathbb{Z}} (-1)^n q^{\frac{3n^2-n}{2}}.$$

In fact $g_n = \frac{3n^2-n}{2}$ is the nth pentagonal number:



$$g_1 = 1 \qquad g_{-1} = 2$$
$$g_2 = 5 \quad \rightsquigarrow \quad g_{-2} = 5$$
$$g_3 = 12 \qquad g_{-3} = 15$$
$$\vdots$$

Note that the Euler function is the inverse of the generating function of partitions

$$\varphi(q)^{-1} = \prod_{n=1}^{\infty} (1-q^n)^{-1} = \prod_{n=1}^{\infty} (1+q^n+q^{2n}+\dots)$$

$$= \sum_{\substack{n \\ 0\le\lambda_1\le\lambda_2\le\dots\le\lambda_n}} q^{\lambda_1+\dots+\lambda_n} = \sum_n p(n)\, q^n$$

where $p(n)$ is the number of partitions of n.

We hence obtain

$$\left(\sum_{n=0}^{\infty} p(n)\, q^n\right)\left(1+\sum_{n=1}^{\infty}(-1)^n\left(q^{g_n}+q^{g_{-n}}\right)\right)=1$$

which yields a linear recurrence for $p(n)$

$$p(n)=\sum_m (-1)^{m+1}\, p(n-g_m), \text{ so}$$

$$p(n)=p(n-1)+p(n-2)-p(n-5)-p(n-7)+p(n-12)+\ldots$$

which terminates, since $p(m)=0$ for $m<0$.

<u>Exercise</u>: Research how to similarly deduce the <u>Rogers-Ramanujan</u> identities.

__Further directions:__ Denominator identities for general (twisted) affine Lie algebras yield the __Macdonald identities__.

Lecture 20

2 Kac-Moody groups

$k = \mathbb{C}$. Variety = locally closed subset of $\mathbb{P}^n_{\mathbb{C}}$

closed $\cap$ open

2.1. Reductive groups in one lecture

Def A: (1) An algebraic group G is a variety, s.t., multiplication, inversion and unit are morphisms,

(2) An alg. grp. G is called linear if it is a closed subgroup of some GL_n and affine if the underlying variety is. $\square$

Lemma A: (1) Affine $\Longleftrightarrow$ Linear

(2) $H \subset G$ affine $\rightsquigarrow$ G/H is a variety

Example A: GL_n, $G_m = GL_1$, SL_n

$G_a = \left\{ \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \right\} \subset GL_2$, SO_n, Sp_n, ...

are lin. alg. groups. $\square$

From now on: G lin. alg. grp.

Def B: The Lie algebra of G is

$$\mathfrak{g} = \mathrm{Lie}(G) = T_e G$$

equipped with a Lie bracket $\square$

<u>Example B</u>

$$\begin{tikzcd} G \arrow[d, squiggly] & < & GL_n \arrow[d, squiggly] \\ \mathfrak{g} & \subset & \mathfrak{gl}_n = k^{n\times n} \end{tikzcd}$$

B

<u>Def C</u> (1) Let $g \in G \subset GL_n$.

Then there are unique $g_u, g_{ss} \in GL_n$, such that

$$g = g_u g_{ss} = g_{ss} g_u,$$

where g_{ss} is diagonalizable, g_u is unipotent ($(g-1)^n = 0$) in GL_n

(2) Let $X \in \mathfrak{g} = \mathrm{Lie}(G) \subset \mathfrak{gl}_n$

Then there are $X_n, X_{ss} \in \mathfrak{gl}_n$, such that

$$X = X_n + X_{ss}, \quad [X_n, X_{ss}] = 0$$

and X_{ss} semisimple, X_n nilpotent in $\mathfrak{gl}_n$.

(3) G is called diagonalizable / unipotent

$g = g_{ss}$ or $g = g_u$ for all $g \in G$.

Similar definition for $\mathfrak{g}$. $\square$

<u>Example C</u>: (1) $(\mathbb{G}_m)^n$ is <u>diagonizable</u>.

$G \cong \mathbb{G}_m^n$ is called a torus

(2) $U = \left\{ \begin{pmatrix} 1 & & * \\ & \ddots & \\ & & 1 \end{pmatrix} \right\}$ is unipotent

$\mathfrak{n} = \mathrm{Lie}\, U = \left\{ \begin{pmatrix} 0 & & * \\ & \ddots & \\ & & 0 \end{pmatrix} \right\}$ is nilpotent

Thm C: (1) $g_u, g_{ss} \in G$, $X_n, X_{ss} \in \mathfrak{g}$.

The decompositions are intrinsical.

(2) There is an equivalence of categories

$$\{\text{unipotent linear alg. grp.'s}\} \to \{\text{f.d. nilpotent Lie algebras}\}$$

$$G \longmapsto \mathrm{Lie}(G)$$

$$\text{"}\exp(\mathfrak{n})\text{"} \longleftarrow \mathfrak{n}$$

(3) Let $X(G) = \mathrm{Hom}_{grp}(G, \mathbb{G}_m)$,

$Y(G) = \mathrm{Hom}_{grp}(\mathbb{G}_m, G)$.

Then there is an equivalence of categories

$\{\text{diagonalizable groups}\} \leftrightarrow \{\text{f.g. abelian grp's}\}$

$$\mathcal{G} \longmapsto X(T)$$

$$\mathrm{Hom}(A, \mathbb{G}_m) \longleftarrow A$$

(4) If $\mathcal{G} = T$ is a torus, $\mathfrak{h} = \mathrm{Lie}\, T$, then $X(T) \subset \mathfrak{h}^*$ and $Y(T) \subset \mathfrak{h}$ are lattices. We obtain:

$$\begin{array}{ccc} \mathrm{Irr}(T) & = & X(T) \\ \downarrow & & \downarrow \\ \mathrm{Irr}(\mathfrak{h}) & = & \mathfrak{h}^* \end{array}$$

(4) If $\mathcal{G}$ is solvable and connected

$$\mathcal{G} = T \ltimes \mathcal{G}_u, \quad \mathcal{G}_u = \{g \mid g = g_u\} \qquad \square$$

<u>Def D</u>: Let G be conn.

(1) The largest normal connected (unipotent) solvable subgroups

$$R_u(G), R(G) \trianglelefteq G$$

are called the <u>(unipotent) radical</u> of G.

G <u>reductive (semisimple)</u> $\Leftrightarrow R_{(u)}(G) = 0$

(2) A maximal conn. diagonalizable $T \subset G$ is called a <u>max. torus</u>, a max. conn. solvable $B \subset G$ a Borel subgroup. $\square$

<u>Sketch</u>: $$\overbrace{R_u(G) \subseteq R(G)}^{\text{torus}} \subseteq G$$

$$\underbrace{R_u(G) \subseteq \underbrace{R(G) \subseteq G}_{\text{semisimple}}}_{\text{reductive}}$$

Thm D: (1) If G is s.s. of rank 1, either

$G \cong PGL_2$ or $G \cong SL_2$

(2) All max. tori and Borel are conjugate.

(3) If G is reductive, $G/Z(G)$ and $[G,G]$

are semisimple. $[G,G] \to G/Z(G)$ is f.t.

(4) $\begin{array}{c} G \\ \cup \\ T \end{array}$ s.s. $\iff$ $\begin{array}{c} \mathfrak{g} \\ \cup \\ \mathfrak{h} \end{array}$ s.s.

T max. torus $\qquad \mathfrak{h} = \mathrm{Lie}(T) =$ Cartan

G red. $\iff$ $\mathfrak{g}$ red.

Example D: $G = GL_n$, $B = \{(\nabla^{*})\}$, $T = \{(\diagdown)\}$.

$[G, G] = SL_n \xrightarrow{n-1:1} PGL_n = G/Z(G)$.

$\mathrm{Lie}(SL_n) = \mathrm{Lie}(PGL_n) = \mathfrak{sl}_n$ $\circlearrowleft$

Def E: let G red. $G \supset B \supset T$ Borel + max torus.

Then T acts on $\mathfrak{g}$ and we may decompose

$$\mathfrak{g} = \underbrace{\bigoplus_{\alpha \in \Delta_-} \mathfrak{g}_\alpha \oplus \overbrace{\mathfrak{h}}^{\mathrm{Lie}(T)} \oplus \bigoplus_{\alpha \in \Delta_+} \mathfrak{g}_\alpha}$$

$$\qquad\qquad\qquad\qquad \underbrace{\mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta_+} \mathfrak{g}_\alpha}_{\mathrm{Lie}(B)}$$

(1) $\Delta_{(\pm)} \subset X(T)$ is the set of (pos./neg.) roots

(2) $\alpha \in \Delta_+$ is called simple if $\alpha \notin \mathbb{Z}_{\geq 0}(\Delta_+ \setminus \{\alpha\})$

$\Pi \subset \Delta_+ \subset \Delta$ is the set of <u>simple roots</u>

(2) $W = N_G(T)/T$ is the <u>Weyl group</u> $\square$

<u>Thm E</u>: (1) Let $\alpha \in \Delta_+$. Set $Z_\alpha = C_G(\ker\alpha)$ and

$G_\alpha = Z_\alpha / Z(Z_\alpha)$. Then G_α is s.s. of rank 1.

There is a unique map

$$SL_2 \longrightarrow Z_\alpha, \quad \text{such that}$$

$SL_2 \twoheadrightarrow G_\alpha$, $\begin{pmatrix} * & * \\ & * \end{pmatrix} \to B$. Denote

$\alpha^\vee : \mathbb{G}_m \xrightarrow{\sim} \left\{ \begin{pmatrix} x & \\ & x^{-1} \end{pmatrix} \right\} \longrightarrow T$. Then $\alpha^\vee \in Y(T)$

is called the coroot of α. (Set $(-\alpha)^\vee = -(\alpha^\vee)$).

(2) Under the embeddings, $X(T) \subset \mathfrak{h}$, $Y(T) \subset \mathfrak{h}^*$, roots and coroots agree with the roots / coroots of the reductive Lie algebra.

(3) $(X(T), \Delta, Y(T), \Delta^\vee, \langle , \rangle, ( )^\vee)$ is called the root datum of G.

(4) $\{\text{red. groups}\}/\cong \xleftrightarrow{1:1} \{\text{root data}\}/\cong$ $\square$

Sketch: (1) Let G be semisimple

$$\begin{tikzcd}
G \supset B \supset T \arrow[r, squiggly] \arrow[dd, squiggly] & (X(T), \Delta, Y(T), \Delta^\vee) \text{ root datum} \\
& + \text{choice of } \Pi, \Pi^\vee \arrow[d, squiggly, "\otimes \mathbb{R}"] \\
\mathfrak{g} \supset \mathfrak{b} \supset \mathfrak{h} \arrow[r, squiggly] \arrow[dd, equal] & (\mathbb{R}X(T), \Delta, \Delta^\vee) \\
& + \text{choice of } \Pi, \Pi^\vee \arrow[d, squiggly, "\otimes \mathbb{C}"] \\
\mathfrak{g}(A) \supset \mathfrak{b} \supset \mathfrak{h} & (\mathfrak{h}, \Pi, \Pi^\vee) \text{ realization} \arrow[l, Leftarrow]
\end{tikzcd}$$

$\mathfrak{b} = \mathfrak{n}^+ + \mathfrak{h}$

(2) $G_{sc} \twoheadrightarrow G \twoheadrightarrow G_{ad}$

G_{sc}: trivial π_1, simply-connected

G_{ad}: trivial center, adjoint

$\square$

Lecture 21

<u>Def F</u>: For each $\alpha \in \Delta$, there is a unique subgroup

$$u_\alpha : \mathfrak{g}_a \to U_\alpha \subset G,$$

s.t. $\operatorname{Lie} U_\alpha = \mathfrak{g}_\alpha$ and for $t \in T$,

$$t u_\alpha(x) t^{-1} = u_\alpha(\alpha(t) x).$$

This is called the <u>root subgroup</u> associated to α.

<u>Thm</u>: Let $U = \operatorname{Rad}_u B = B_u \subset B$ and $U' \subset U$ a T-invariant connected subgroup. Then, multiplication yields an isomorphism of varieties

$$\prod_{\substack{\alpha \in \Delta_+ \\ U_\alpha \subset U}} U_\alpha \longrightarrow U' \qquad \square$$

2.1 Ind-Varieties

Def A (1) An <u>ind-variety</u> X is a set with increasing filtration

$$X_0 \subseteq X_1 \subseteq X_2 \subseteq \dots$$

(a) $X = \bigcup_{n \geq 0} X_n$ and

(b) each X_n is a variety and $X_n \hookrightarrow X_{n+1}$ a closed embedding

(2) We call X affine / projective, if each X_n is.

(3) Write $k[X] = \lim_n k[X_n]$ and equip it with the inverse limit topology.

(4) A morphism of ind-varieties is a map $f: X \to Y$, s.t.,

for all $n \geq 0$, there is an $m(n) \geq 0$, s.t. $f(X_n) \to Y_{\leq m(n)}$

is a morphism of varieties and the map

$f^\#: k[Y] \to k[X]$ is continuous. $\square$

Similarly one defines closed embeddings,

open/closed subsets, the Zariski topology and

the structure sheaf $\mathcal{O}_X$.

Examples: (1) The infinite affine space

$\mathbb{A}^\infty = \{(a_i)_{i=1}^\infty \mid \text{almost all } a_i = 0\}$

is an ind-variety, filtered by $\mathbb{A}^1 \subset \mathbb{A}^2 \subset \dots$.

(2) Similarly any vector space V of countable dimension is an ind-variety, and so is $\mathbb{P}(V)$ and any set homogeneous elements in $\hat{S}(V) = k[V]$ defines a projective ind-variety in $\mathbb{P}(V)$

(3) Let $S = \{s_1, s_2, ..\}$ a countably infinite set. Then S is an ind-variety filtered by the discrete varieties $S_n = \{s_1, .., s_n\}$ $\square$

<u>Def B</u>: For a point in an ind-variety $x \in X$, the tangent space at x is

$$T_x(X) = \operatorname{colim} T_x(X_n)$$

For a morphism $f : X \to Y$ we obtain a map

$$(df)_x : T_x X \to T_x Y.$$

$\square$

Def C: (1) An ind-group G is an ind-variety, with multiplication, inversion and unit morphisms of ind-varieties.

(2) G is affine if the underlying variety is

(3) An algebraic representation V is a vector space of countable dimension, s.t., the action map $G \times V \to V$ is a morphism of ind-varieties. $\square$

From now let $\mathcal{G}$ be an affine ind-group.

<u>Thm</u>: The tangent space at the identity $T_e\mathcal{G}$ has the structure of a Lie algebra, denoted by $\mathrm{Lie}\,\mathcal{G}$. Any algebraic rep. yields a representation of $\mathrm{Lie}\,\mathcal{G}$ by differentiation. $\square$

<u>Rmk</u>: (1) There is a natural notion of vector bundles, Picard group, cohomology, ... for ind-varieties

(2) There are different notion of smoothness of ind-varieties. For example a point $x \in X$, s.t. $x \in X_m$ is smooth $\forall m > n$, would be regarded

a smooth point of X. Hence, $\mathbb{A}^\infty$ and $\mathbb{P}(V)$ are smooth $\square$

2.2 Pro-groups and Pro-Lie Algebras

<u>Def</u> A (1) A pro-group $\mathcal{G}$, is a group together with a defining set $\mathcal{F}$ of normal subgroups of $\mathcal{G}$, and the structure of an affine variety on $\mathcal{G}/N$ for each $N \in \mathcal{F}$, s.t.,

(a) $N_1 \cap N_2 \in \mathcal{F} \quad \forall N_1, N_2 \in \mathcal{F}$

(b) If $\quad \mathcal{G} \trianglerighteq N \trianglerighteq M$ and $M \in \mathcal{F}$, then $N \in \mathcal{F}$

$\quad \Leftrightarrow N/M \subseteq \mathcal{G}/M$ is closed

(c) $\mathcal{G}/N_1 \to \mathcal{G}/N_2$ is a morphism of varieties $\forall N_1 \subset N_2 \in \mathcal{F}$.

(d) The map $\mathcal{G} \to \lim\limits_{N \in \mathcal{F}} \mathcal{G}/N$ is a bijection.

(2) A pro-group morphism $\varphi: \mathcal{G} \to \mathcal{G}'$ is

a group morphism, s.t. $\forall N' \in \mathcal{F}'$, $\varphi^{-1}(N') \in \mathcal{F}$ and $\mathcal{G}/\varphi^{-1}(N) \to \mathcal{G}'/N$ is a morphism of algebraic groups.

(2) The pro-topology on $\mathcal{G}$ is the inverse limit topology on $\mathcal{G} \cong \varprojlim_{N \in \mathcal{F}} \mathcal{G}/N \subset \prod_{N \in \mathcal{F}} \mathcal{G}/N$. $\square$

<u>Remark A</u>: Let $\{\mathcal{G}_i\}_{i \geq 0}$ a family of alg. grp's with surjective maps $\mathcal{G}_{i+1} \to \mathcal{G}_i$. Then the limit

$$\mathcal{G} = \varprojlim_{i \geq 0} \mathcal{G}_i$$

in the category of groups is a pro group with

$\mathcal{F} = \{ \pi_i^{-1}(N) \mid N \leq \mathcal{G}_i$ is a normal closed subgroup$\}$

where $\pi : \mathcal{G} \to \mathcal{G}_i$ is the natural map. $\square$

There is a notion of (normal) subgroups, quotients and isomorphism thm. for pro-groups.

Def B (1) A <u>pro-Lie algebra</u> $\mathfrak{g}$ is a Lie algebra over k, with a defining set $\mathcal{F}$ of ideals of $\mathfrak{g}$ of finite codimension, s.t.,

(a) $\mathfrak{c}_1 \cap \mathfrak{c}_2 \in \mathcal{F}$ $\forall\, \mathfrak{c}_1, \mathfrak{c}_2 \in \mathcal{F}$

(b) $\mathfrak{g} \supset \mathfrak{j} \supset \mathfrak{c}$ for an ideal $\mathfrak{j}$ and $\mathfrak{c} \in \mathcal{F}$ $\Rightarrow$ $\mathfrak{j} \in \mathcal{F}$

(c) The map $\mathfrak{g} \to \varprojlim_{\mathfrak{c} \in \mathcal{F}} \mathfrak{g}/\mathfrak{c}$ is an isomorphism.

(2) A morphism .... $\square$

Remark B: Let $\{g_i\}$ be a family of f.d. Lie alg. with surj. maps $g_{i+1} \to g_i$. Then

$$g = \lim_{i} g_i$$

is a pro-Lie algebra with $\mathcal{F} = \{\pi_i^{-1}(a_i) \mid a_i \subseteq g_i$ is an ideal$\}$.

Example: (1) Let V be vector with a filtration

$$S = V_0 = (0) \subset V_1 \subset V_2 \subset \cdots$$

s.t. V_i is finite-dimensional and $V = \bigcup V_i$. Let

$$u_n(S) = \{ f \in \operatorname{End} V_n \mid f(V_i) \subset V_{i-1} \ \forall i \}$$ and

$$U_n(S) = \{ f \in GL V_n \mid (f-1)(V_i) \subset V_{i-1} \ \forall i \}.$$

Let $\mathfrak{u}(S) = \mathfrak{u}_\infty(S) \subset \mathrm{End}(V)$ and

$U(S) = U_\infty(S) \subset GL(V)$. Then the restriction maps

$\mathfrak{u}_i(S) \to \mathfrak{u}_{i-1}(S)$ and $U_i(S) \to U_{i-1}(S)$ yield

$$\mathfrak{u}(S) = \lim \mathfrak{u}_i(S), \quad U(S) = \lim U_i(S)$$

which are hence a pro-Lie algebra / group

Lecture 22

(2) Let G be a reductive group, then

$$L^+G = G(k[[t]]) = \lim_n G(k[[t]]/t^n)$$

is a pro-group called the positive loop group with

Lie algebra $\mathfrak{g} \otimes k[[t]]$ $\square$

<u>Def 5</u>: Let $\mathcal{G}$ be pro-group with defining set $\mathcal{F}$.

(1) For $N \in \mathcal{F}$, let $\mathfrak{g}_N = \mathrm{Lie}\ \mathcal{G}/N$. Then the <u>pro-Lie algebra</u> of $\mathcal{G}$ is

$$\mathrm{Lie}(\mathcal{G}) = \lim_{N \in \mathcal{F}} \mathfrak{g}_N$$

(2) Recall that $k = \mathbb{C}$. For $N \in \mathcal{F}$, let

$$\exp_N : \mathfrak{g}_N \to \mathcal{G}/N$$

be the <u>exponential map</u>. Then we obtain the (pro-) exponential map

$$\exp : \mathfrak{g} \to \mathcal{G}$$

in the limit. $\square$

Def D: (1) A pro-group $\mathcal{G}$ is called pro-unipotent if all $\mathcal{G}/N$ for $N \in \mathcal{F}$ are unipotent.

A pro-Lie algebra $\mathfrak{g}$ is called pro-nilpotent if ...

(2) Let $\mathcal{G}$ be a pro-group. A pro-representation V is a rep., s.t. any $v \in V$ is contained in a f.d. subrep. $W \subset V$ with

(a) $\exists N \in \mathcal{F}$, s.t. N acts trivial on W

(b) the rep. of $\mathcal{G}/N$ on W is algebraic

Similar for $\mathfrak{g}$ .... $\square$

<u>Thm.</u>: (1) The functor Lie(−) yields an equivalence of categories

$$\{\text{unipotent pro-groups}\} \to \{\text{nilpotent pro-Lie algebras}\}$$

(2) Let $\mathcal{G}$ be a pro-unipotent pro-group and $\mathfrak{g} = \operatorname{Lie} \mathcal{G}$. Then differentiation yields an equivalence of categories

$$\{\text{pro-rep's of } \mathcal{G}\} \to \{\text{locally nilpotent pro-rep's of } \mathfrak{g}\}$$

each $x \in \mathfrak{g}$ act locally nilpotently. $\square$

2.3 Tits Systems

<u>Def A:</u> A Tits system is a tuple

$$(G, B, N, S)$$

where G is a group with subgroups B, N and $S \subset W = N/T$ for $T = B \cap N$, such that:

(1) $B \cup N$ generates G, $B \cap N \trianglelefteq N$

(2) S generates $W = N/(B \cap N)$ and $|s| = 2$ $\forall s \in S$

(3) $sBw \subset BwB \cup BswB$ for $s \in S$, $w \in W$.

(4) $sBs \not\subset B$ for all $s \in S$. $\square$

For $w \in W$, we denote $BwB = BnB$, where $nT = w$

Example: Let $G \supset B$ be a reductive group with Borel subgroup. Then

$$(G, B, N = N_G(T), S = \{s_i\}) \quad \text{simple reflections}$$

is a Tits system $\square$

Prop: Let $s \in S$, $w, w' \in W$ and $\ell = \ell_S : W \to \mathbb{Z}_{\geq 0}$

$w \mapsto \min\{n \mid w = s_{i_1} \cdots s_{i_n},\ s_{i_j} \in S\}$ the length function. Then

(1) $BwB\,Bw'B \subset Bww'B$, $Bw^{-1}B = (BwB)^{-1}$

(2)
$$BsB \cdot BwB = \begin{cases} BswB & \text{if } \ell(sw) = \ell(w)+1 \\ BwB \cup BswB & \text{if } \ell(sw) = \ell(w)-1 \end{cases}$$

(3) $BsB \neq B$, $s^2 = 1$

(4) For $s_1, \ldots, s_n \in S$, we get

$$B s_1 \cdots s_n B w B \subset \bigcup_{1 \le i_1 < \cdots < i_m \le p} B s_{i_1} \cdots s_{i_m} w B$$

Proof: Exercise $\square$

Thm A (a) For $I \subset S$, let $W_I = \langle s \in I \rangle$, $P_I = B W_I B$

Then $P_I \subset G$ is a subgroup, called parabolic subgroup

(b) $G = BWB = \bigoplus_{w \in W} BwB$. This is called the

Bruhat decomposition of G.

Pf: (a) Use previous Prop (4)

(b) By (a) $BWB \subset G$ is a subgroup. Since $N, B \subset BWB$

generate G, $BWB = G$. It remains to show

$w \neq w' \Rightarrow BwB \neq Bw'B$. W.l.o.g. $\ell(w) \geq \ell(w') = n$.

We show the statement by induction on n.

If $q = 0$, $w' = e$ and $Bw'B = B \neq BwB$.

If $q \geq 1$, choose $s \in S$, s.t., $\ell(sw') = q-1$.

Hence $\ell(w) > \ell(sw')$ and $w \neq sw'$ and $sw \neq sw'$

and $\ell(sw) \geq \ell(w) - 1 \geq \ell(sw') = q-1$.

By induction, $Bsw'B \neq BwB, BswB$, so

$$Bsw'B \cap BsBBwB = Bsw'B \cap (BwB \cup BswB) = \emptyset$$

which shows, by moving BsB $\cap$:

$$Bw'B \cap BwB \subset BsBBsw'B \cap BwB = \emptyset \qquad \square$$

Example: let $G = GL_n$, $B = \{(\text{upper triangular})\}$, $w_0 = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}$.

Then $B^- = w_0 B w_0 = \{(\text{lower triangular})\}$ and

$$G = \biguplus_{w \in W} B^- w B$$

is the LR-decomposition ∎

Rem: For $I, J \subset W$, one obtains

$$G = \biguplus_{w \in W_I \backslash W / W_J} P_I w P_J$$ ∎

Thm B: (W, S) is a Coxeter group, that is, there are $m_{s,t} \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$, s.t.,

$$W = \langle s \in S \mid (st)^{m_{s,t}} = 1 \rangle$$ ∎

Lecture 23

2.4 Tits systems and almagamation

Def: (1) Let I be a set and $\mathcal{G}_i$, $\mathcal{G}_{\{i,j\}} = \mathcal{G}_{\{j,i\}}$ groups with morphisms $\varphi_{ij} : \mathcal{G}_{\{i,j\}} \to \mathcal{G}_i$, $\varphi_{ji} : \mathcal{G}_{\{i,j\}} \to \mathcal{G}_j$.

Then the amalgamated product $\underset{\mathcal{G}_{\{i,j\}}}{*} \mathcal{G}_i$ is the colimit (in the category of groups) of the diagram

$$\begin{tikzcd} & \mathcal{G}_{\{i,j\}} \arrow[dl] \arrow[dr] & \cdots \\ \mathcal{G}_i & & \mathcal{G}_j \quad \cdots \end{tikzcd}$$

(2) If all $\mathcal{G}_i \subset \mathcal{G}$ and $\mathcal{G}_{\{i,j\}} = \mathcal{G}_i \cap \mathcal{G}_j$, we say that $(\mathcal{G}_i)_{i \in I}$ is a system of groups. $\square$

Example: let $G = \langle \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \rangle$, $H = \langle \begin{pmatrix} 1 & -1 \\ 1 & \end{pmatrix} \rangle \subset SL_2(\mathbb{Z})$.

$$\begin{array}{cc} \text{\rotatebox{90}{$\cong$}} & \text{\rotatebox{90}{$\cong$}} \\ \mathbb{Z}/4 & \mathbb{Z}/6 \end{array}$$

Then $G \cap H = \langle \begin{pmatrix} -1 & \\ & -1 \end{pmatrix} \rangle \cong \mathbb{Z}/2$ and

$$SL_2(\mathbb{Z}) \cong G \underset{G \cap H}{*} H \cong \mathbb{Z}/4 \underset{\mathbb{Z}/2}{*} \mathbb{Z}/6 \qquad \square$$

Thm A: If (G, B, N, S) is a Tits system G is the almagamated product of the system of subgroups

$\{N, P_S, s \in S\}$ $\qquad$ B

Proof Sketch: Let G' be the alm. prod. Then we obtain a map $G' \to G$ which induces isomorphisms $N' \to N$, $P_S' \to P_S$, $B \to B$ and $W' = N'/(B' \cap N') \to W$.

One shows that $(\mathcal{G}', B', N', S)$ is a Tits system.

Hence $\mathcal{G}' = \biguplus B'wB'$, and $\varphi(B'wB') = BwB$.

Hence, if $g \in B'wB' \cap \ker\varphi$, $w = 1$, so

$g' \in B' \cap \ker\varphi = \{1\}$. Hence $\varphi : \mathcal{G}' \xrightarrow{\sim} \mathcal{G}$ $\square$

We now explain how to recover $\mathcal{G}$ from $\{B, N, P_s, s \in S\}$

without using that they are subgroups of $\mathcal{G}$.

For this consider the following data:

- a finite set S, $(B, N, P_s; s \in S)$ a system of groups
- $Y = N \cup \bigcup_{s \in S} P_s$, $T = B \cap N$, $N_s = N \cap P_s$, $W = N/T$, $\pi : N \to W$
- Assume that $B \subset P_s \ \forall s$.

- Let $n \in N$ with $n = n_1 \cdots n_r$ for $n_i \in N_{s_i}$ and define $B(n_1, \ldots, n_r)$ inductively by

  - $B(n_1) = B \cap n_1^{-1} B n_1$
  - $B(n_1, \ldots, n_r) = B \cap n_r^{-1} B(n_1, \ldots, n_{r-1}) n_r$.

- Define the map

$$\gamma(n_1, \ldots, n_r) : B(n_1, \ldots, n_r) \to B, \quad x \mapsto n x n^{-1}$$

<u>Thm B</u>: Assume that the above system satisfies

(1) $P_s \cap P_{s'} = B$ for $s \neq s'$ (2) $T \trianglelefteq N$

(3) For $s \in S$, $N_s / T = \{1, s\}$ of order 2.

(4) $P_s = B \cup BsB$

(5) (W, S) is a Coxeter system

(6) If $n = n_1 \cdots n_r$ and $n_i \in N_{s_i}$ such that

$w = \pi(n) = \pi(n_1) \cdots \pi(n_r)$ is a reduced expression,

$\gamma_w : B_w = B(n_1, \ldots, n_r) \subset B$ depends only on w.

(7) For $w \in W$, $s \in S$ with $\ell(sw) > \ell(w)$, $B_w \cdot B_s = B$.

(8) Let $s, t \in S$, $w \in W$, s.t., $sw = wt$, $\ell(sw) > \ell(w)$. Then for

any $m \in \pi^{-1}(s)$, $n \in \pi^{-1}(w)$, $b \in B \setminus B_t$, there is a

$y \in (bB_t) \cap B$ and $y', y'' \in B_w$ such that for

$m' = n^{-1} m^{-1} n$ one gets:

- $(m')^{-1} y m' = y' m' y'' \in P_t$

- $m \gamma_w(y) m^{-1} = \gamma_n(y') m^{-1} \gamma_n(y'') \in P_s$

(9) B is not normal in P_s.

Let $\mathcal{G}$ be the alm. prod. of $(B, N, P_s, s \in S)$. Then $Y \hookrightarrow \mathcal{G}$ is injective, $(\mathcal{G}, B, N, S)$ is a Tits system. For any group $\mathcal{G}'$ with an injective map $Y \to \mathcal{G}'$ which is a group hom. on N and P_s, and $\mathcal{G}'$ is generated by Y, then $\mathcal{G} \xrightarrow{\sim} \mathcal{G}'$ is an isomorphism $\square$

2.5 Construction of T and N

Let $(\mathfrak{h}, \Pi, \Pi^\vee)$ be a realization of a GCM A.

We choose an integral form $\mathfrak{h}_\mathbb{Z} \subset \mathfrak{h}$, s.t.

(1) $\mathfrak{h}_\mathbb{Z} \otimes \mathbb{C} = \mathfrak{h}$

(2) $\alpha_i^\vee \in \mathfrak{h}_\mathbb{Z}$

(3) $\alpha_i \in \mathfrak{h}_\mathbb{Z}^* = \mathrm{Hom}(\mathfrak{h}_\mathbb{Z}, \mathbb{Z}) \subset \mathfrak{h}^*$

(4) $\mathfrak{h}_\mathbb{Z} / \sum \alpha_i^\vee$ is torsion free.

<u>Remark</u>: $\mathcal{D} = (\mathfrak{h}_\mathbb{Z}^*, \Pi, \mathfrak{h}_\mathbb{Z}, \Pi^\vee)$ is a

free, cofree, cotorsion free <u>Kac-Moody root datum</u>.

$\{\alpha_i\}$ l.i. $\quad \{\alpha_i^\vee\}$ l.i. $\quad \mathfrak{h}_\mathbb{Z}/\sum \alpha_i^\vee$ torsion-free

The construction (with slight adjustments) also work for general Kac-Moody root data $\square$

We define the <u>maximal torus</u> by

$$T = \mathrm{Hom}_{\mathbb{Z}}(\mathfrak{h}_{\mathbb{Z}}, \mathbb{G}_m).$$

Then, as explained in 2.1 Thm C, we get

$$X(T) = \mathrm{Hom}(T, \mathbb{G}_m) = \mathfrak{h}_{\mathbb{Z}}^*$$

$$Y(T) = \mathrm{Hom}(\mathbb{G}_m, T) = \mathfrak{h}_{\mathbb{Z}}$$

and $\mathrm{Lie}(T) = \mathfrak{h}$. In particular, every coroot $\alpha_i^\vee \in \mathfrak{h}_{\mathbb{Z}} = Y(T)$, defines a map $\alpha_i^\vee : \mathbb{G}_m \to T$.

Also every weight module of $\mathfrak{h}$ with weights $P(V) \subset \mathfrak{h}_{\mathbb{Z}}^*$ integrates to a rep. of T.

Now we define a group N with $N/T = W$.

Recall that $W = \langle r_i, i \in I \mid r_i^2 = 1, (r_i r_j)^{m_{ij}} = 1 \rangle$ for

some m_{ij} depending on a_{ij}, a_{ji}, see 1.10 Prop.

Let N be the group generated by its subgroup

T and the set $\{\tilde{r}_i \mid i \in I\}$ with relations

- $\tilde{r}_i t \tilde{r}_i^{-1} = r_i(t) \in T$
- $(\tilde{r}_i)^2 = \alpha_i^\vee(-1) \in T$
- $\tilde{r}_i \tilde{r}_j \tilde{r}_i \cdots = \tilde{r}_j \tilde{r}_i r_j \cdots$ with m_{ij} factors on each side.

Lemma: Let V be an integrable rep. of $\mathfrak{g}(A)$ with weights $P(V) \subset \mathfrak{h}_{\mathbb{Z}}^*$. Then, the action of T on V extends to an action of N, where $\tilde{r}_i$ acts via

$$(\exp f_i)(\exp(-e_i))(\exp f_j)$$

$\square$

Prop: The map $T \cup \{\tilde{r}_i\} \to N$ is inj. and there is a s.e.s. $1 \to T \to N \to W \to 1$

$$\tilde{r}_i \mapsto r_i$$

$\square$

Lecture 24

2.6 Construction of $\mathcal{U}$.

Recall that $\mathfrak{g} = \mathfrak{g}(A) = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ and

$$\mathfrak{n}_+ = \bigoplus_{\alpha \in \Delta_+} \mathfrak{g}_\alpha .$$

Define the completion of $\mathfrak{n}_+$ and $\mathfrak{g}$ via

$$\hat{\mathfrak{n}} = \prod_{\alpha \in \Delta_+} \mathfrak{g}_\alpha \quad , \quad \hat{\mathfrak{g}} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \hat{\mathfrak{n}}$$

These are Lie algebras via the obvious formulas

$$\left[\sum_{\alpha \in \Delta_+} x_\alpha , \sum_{\alpha \in \Delta_+} y_\alpha\right] = \sum_{\gamma \in \Delta_+} \underbrace{\sum_{\substack{\alpha+\beta=\gamma \\ \alpha,\beta \in \Delta_+}}}_{\text{finite}} [x_\alpha, x_\beta] \quad \text{and}$$

$$\left[\sum_{\alpha \in \Delta_+} x_\alpha , \underbrace{y}_{\in\, \mathfrak{n}^- \oplus \mathfrak{h}}\right] = \sum_{\alpha \in \Delta_+} [x_\alpha, y]$$

Exercise: Why is this welldefined?

We can give $\hat{n}_+$ the structure of a pro-Lie algebra.

Recall the principal = height grading on $\mathfrak{g}$:

$$\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}(i) \qquad \mathfrak{g}(i) = \bigoplus_{\substack{\alpha \in Q \\ \mathrm{ht}\,\alpha = i}} \mathfrak{g}_\alpha .$$

This restricts to a grading

$$n_+ = \bigoplus_{i \in \mathbb{Z}_{\geq 1}} \mathfrak{g}(i),$$

For $k \geq 1$, let

$$n_+(k) = \bigoplus_{i \geq k} \mathfrak{g}(i).$$

Then $n_+(k) \lhd n_+$ is an ideal and

$$n_+ / n_+(k) = \bigoplus_{i=1}^{k-1} \mathfrak{g}(i)$$

is f.d. Moreover

$$\hat{n} = \lim_{k} n_+ / n_+(k)$$

so that $\hat{n}_+$ obtains the structure of a pro-Lie algebra by 2.2. Rmk B.

We now define U as the unipotent pro-group associated to the pro-Lie algebra $\hat{n}_+$, see 2.2. Thm.

2.7. The Shape of U

<u>Def. A</u>: A subset $\Theta \subset \Delta_+$ is bracket closed (a bracket-ideal) if $\alpha + \beta \in \Theta$ for all $\alpha, \beta \in \Theta$ (for all $\alpha \in \Theta, \beta \in \Delta_+$).

Θ is called bracket-coclosed if Θ is bracket-closed.

<u>Lemma</u> (1) For $x \in \hat{n}$, $\mathbb{C}x \subset \hat{n}$ is a pro-Lie subalgebra.

(2) For $\Theta \subset \Delta_+$ bracket closed,

$$\hat{n}_{\Theta} = \prod_{\alpha \in \Theta} g_\alpha$$

is a pro-Lie subalgebra of $\hat{n}$ $\square$

Def B: (1) For $x \in \hat{n}$, $\mathcal{U}_x = \exp(\mathbb{C}x)$ is called a one-parameter subgroup.

(2) For $\Theta \subset \Delta_+$ bracket closed, $\mathcal{U}_\Theta = \exp(\hat{n}_\Theta)$ is the pro-unipotent subgroup of $\mathcal{U}$ with pro Lie algebra $\hat{n}_\Theta$ $\square$

Thm: (1) If $\Theta \subset \Delta_+$ is a bracket ideal. Then $\hat{n}_\Theta \subset \hat{n}$ is an ideal and $U_\Theta \subset \mathcal{U}$ is a normal subgroup

(2) If $\Theta \subset \Delta_+$ is bracket-closed and coclosed multiplication yields an isomorphism

$$m : U_\Theta \times U_{\Delta_+ \setminus \Theta} \to U$$

Proof: (1) Use $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$

(2) Since $\hat{n}_\Theta \cap \hat{n}_{\Delta^+ \setminus \Theta} = \{0\}$, $U_\Theta \cap U_{\Delta^+ \setminus \Theta} = \{1\}$.

Hence, m is injective. Let H be the map, s.t.

$$\begin{tikzcd} \hat{n}_\Theta \times \hat{n}_{\Delta_+ \setminus \Theta} \arrow[r, "H"] \arrow[d, "\wr"', "\exp \times \exp"] & \hat{n} \arrow[d, "\wr"', "\exp"] \\ U_\Theta \times U_{\Delta_+ \setminus \Theta} \arrow[r, "m"] & U \end{tikzcd}$$

commutes. It suffices to show that H is surjective.

Let $x = \sum_{\beta \in \Delta_+} x_\beta \in \hat{n}$. We show by induction on k

that there are $y_k \in \hat{n}_\Theta$ and $z_k \in \hat{n}_{\Delta_+ \setminus \Theta}$, s.t.,

(a) $(y_k)_\beta = (z_k)_\beta = 0$ for $\operatorname{ht}\beta > k$

(b) $H(y_k, z_k)_\beta = x_\beta$ for $\operatorname{ht}\beta \leq k$

(c) $(y_k)_\beta = (y_{k-1})_\beta, (z_k) = (z_{k-1})_\beta$ for $\operatorname{ht}\beta < k$.

For $k=0$, let $y_0 = z_0 = 0$. Set

$$y_{k+1} = y_k + \sum_{\substack{\beta \in \oplus \\ \operatorname{ht}\beta = k+1}} (x_\beta - H(y_k, z_k)_\beta)$$

$$z_{k+1} = z_k + \sum_{\substack{\beta \in \Delta_+ \setminus \oplus \\ \operatorname{ht}\beta = k+1}} (x_\beta - H(y_k, z_k)_\beta).$$

These clearly fulfill (a) and (c).

For (b), recall the Campbell-Hausdorff formula:

$H(X,Y) = \log(\exp X \cdot \exp Y) =$

$$\sum_{n>0} \frac{(-1)^{n-1}}{n} \sum_{\substack{r_i+s_i>0 \\ 1\le i\le n}} \frac{\left(\sum_{i=1}^{n}(r_i+s_i)\right)^{-1}}{r_1!s_1!\cdots r_n!s_n!} \operatorname{ad}(X)^{r_1}\operatorname{ad}(Y)^{s_1}\cdots\operatorname{ad}(X)^{r_n}\operatorname{ad}(Y)^{s_n-1}(Y)$$

Now, compute $H(y_{k+1}, z_{k+1})$. This has the type of terms:

(i) y_k, z_k only: they yield $H(y_k, z_k)$

(ii) linear terms: $y_{k+} + z_{k+1}$ (contains linear part of $H(y_k, z_k)$)

(iii) other terms involving y_k or z_k <u>and</u> terms of the form $(x_\beta - H(y_k,z_k)_\beta)$

The terms of type (iii) are of $ht \geq k+2$, so irrelevant! In total

$$H(y_{k+1}, z_{k+1}) = H(y_k, z_k) + \sum_{\beta \in \Delta_+} (x_\beta - H(y_k, z_k)_\beta)$$

$$+ \ ht \geq 2$$

which fulfills (ii). Now

$$H(\lim_{k\to\infty} y_k, \lim_{k\to\infty} z_k) = x \qquad \square$$

<u>Example</u>: (1) For $\alpha \in \Delta_+^{re}$, $\{\alpha\}$ is bracket closed and $U_x = U_\alpha$ for $x \in \mathfrak{g}_\alpha \setminus \{0\}$

(2) For $\alpha \in \Pi$, $\Delta_+ \setminus \{\alpha\}$ is a bracket ideal and bracket coclosed and $U = U_{\Delta_+ \setminus \{\alpha\}} \rtimes U_\alpha$.

(3) For $\Delta_w = \Delta_+ \cap w(\Delta_-)$, $|\Delta_w| = \ell(w)$,

Δ_w is bracket closed + coclosed and $\hat{n}_{\Delta_w}$

is f.d., U_{Δ_w} is an algebraic group $\square$

2.8 Constructing Parabolics

Let $X \subset \Pi$. Then we define

- $\Delta_X = \Delta \cap \sum_{\alpha \in X} \mathbb{Z}\alpha$, $\Delta_{X,\pm} = \Delta_X \cap \Delta_{X,\pm}$
- $\mathfrak{g}_X = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta_X} \mathfrak{g}_\alpha$,
- $\mathfrak{u}_X^\pm = \bigoplus_{\alpha \in \Delta_\pm \setminus \Delta_{X,\pm}} \mathfrak{g}_\alpha$, $\mathfrak{u}_X = \mathfrak{u}_X^+$ and
- $\mathfrak{p}_X^\pm = \mathfrak{g}_X \oplus \mathfrak{u}_X^\pm$, $\mathfrak{p}_X = \mathfrak{p}_X^+$

It is easy to show that these are all Lie-algebras and that $\mathfrak{u}_X^\pm \subset \mathfrak{p}_X^\pm$ is an ideal.

Def: We call X of finite type if

$$\mathfrak{g}_X \text{ is f.d.} \iff \det(A|_{X \times X}) \neq 0 \qquad \square$$

Very similarly to $\hat{n}$,

$$\hat{u}_X = \prod_{\alpha \in \Delta_+ \setminus \Delta_{X,+}} g_\alpha$$

is a pro Lie-algebra, and we define

$$\hat{p}_X = g_X \oplus \hat{u}_X .$$

Now, let X be of finite type, then g_X is a reductive Lie algebra and there is a reductive group G_X with maximal torus T, s.t. $\mathrm{Lie}\, G_X = g_X$, see 2.1.

The adjoint action of g_X on $\hat{u}_X$ respects the pro-Lie algebra structure. This implies that the action can be integrated to $\mathcal{G}_X$, so we obtain a map

$$\mathcal{G}_X \to \mathrm{Aut}(\hat{\mathfrak{u}}_X) = \mathrm{Aut}(\mathcal{U}_X)$$

where $\mathcal{U}_X = \mathcal{U}_{\Delta_+ \setminus \Delta_{X,+}} = \exp(\hat{\mathfrak{n}}_X)$.

We define the parabolic subgroup associated to X by

$$\mathcal{P}_X = \mathcal{G}_X \ltimes \mathcal{U}_X .$$

In fact, $\mathcal{P}_X$ inherits the progroup structure from $\mathcal{U}_X$ and $\mathrm{Lie}(\mathcal{P}_X) = \hat{\mathfrak{p}}_X$.

Also, for $X_1 \subset X_2$ of finite type

$$\mathcal{P}_{X_1} \subset \mathcal{P}_{X_2} .$$

Lecture 25

2.9 Kac-Moody group - Definition

We now build $\mathcal{G}$ as an amalgamated product of the subgroups that we constructed in 2.5-2.8.

For $\alpha \in \Pi$, we have

$$N_\alpha \xhookrightarrow{\theta_\alpha} \mathcal{G}_\alpha \subset Q_\alpha \xhookleftarrow{\gamma_\alpha} B$$

where $N_\alpha = \langle T, \tilde{S}_\alpha \rangle = T \uplus \tilde{S}_\alpha T$ and

$$\theta_\alpha(\tilde{S}_\alpha) = \exp(f_\alpha)\exp(-e_\alpha)\exp(f_\alpha) \in \mathcal{G}_\alpha .$$

and $\theta_\alpha|_T = id$.

Now let Z be the quotient of the disjoint union

$$N \uplus \biguplus_{\alpha \in \Pi} Q_\alpha$$

by the equivalence relation generated by

- $\chi_\alpha(b) = \chi_\beta(b)$ $\forall \alpha, \beta \in \Pi$ and $b \in B$

- $n \sim \theta_\alpha(n)$ for all $n \in N_\alpha$, $\alpha \in \Pi$

So B, N, Q_α all inject in Z and $B \cap N = T$ in Z.

<u>Def</u>: The <u>Kac-Moody group</u> $\mathcal{G}$ associated to the root datum $(\mathfrak{h}^*_{\mathbb{Z}}, \Pi, \mathfrak{h}_{\mathbb{Z}}, \Pi^\vee)$ with Cartan matrix A is the almaganated product of the system of groups $(N, Q_\alpha, \alpha \in \Pi)$ in Z. $\square$

2.10 Kac-Moody group - Tits system

<u>Thm</u>: (1) The canonical map $\mathcal{Z} \to \mathcal{G}$ is injective, so $Q_\alpha \hookrightarrow \mathcal{G}$ and $N \hookrightarrow \mathcal{G}$

(2) Let $s_\alpha = \tilde{s}_\alpha T \in N/T$ and $S = \{s_\alpha, \alpha \in \Pi\}$. Then $(\mathcal{G}, B, N, S)$ is a Tits system.

<u>Proof</u>: We apply 2.4 Thm. B, which has <u>nine</u>! assumptions.

<u>(1)</u> For $s \neq s'$, $P_s \cap P_{s'} = B$ ✓

<u>(2)</u> $T \trianglelefteq N$ ✓

<u>(3)</u> For $s \in S$, $|N_s/T| = 2$ ✓

(4) $Q_s = B \cup BsB$:

Use that $\mathcal{G}_\alpha = B' \cup B'sB'$ for $B' = \mathcal{G}_\alpha \cap B$

by 2.3 Thm A + Example. Since

$Q_s = \mathcal{G}_\alpha \ltimes U_{\Delta_+ \setminus \{\alpha\}}$, the statement follows.

(5) (W, S) is a Coxeter system ✓

For conditions (6)–(9), recall that

For $n = n_1 \cdots n_r$ for $n_i \in N_{s_i}$, $B(n_1, \ldots, n_r)$ is defined as

- $B(n_1) = B \cap n_1^{-1} B n_1$ • $B(n_1, \ldots, n_r) = B \cap n_r^{-1} B(n_1, \ldots, n_{r-1}) n_r$,

and we consider the map

$$\gamma(n_1, \ldots, n_r) : B(n_1, \ldots, n_r) \to B, \; x \mapsto n_1(\cdots(n_r x n_r^{-1}) \cdots) n_1^{-1}.$$

(b) Let $n = n_1 \cdots n_r$, s.t., $\pi_1(n_1) \cdots \pi_1(n_r) = \pi(n) = w$ is a reduced expression, then $B(n_1, \dots, n_r) = B_w$ only depend on w and $\gamma(n_1, \dots, n_r) = \gamma_n$ only depends on n:

First, note that for $n \in T\tilde{S}_\alpha$ with $\alpha \in \Pi$ and $\Theta \subset \Delta_+$ bracket closed

$$(*) \quad U \cap n^{-1} U_\Theta n = U_{s_\alpha(\Theta) \cap \Delta_+}.$$

This follows from the analogous statement for $\hat{n}$.

Now write $n = n_1 \cdots n_r$ s.t. $\pi(n_1) \cdots \pi(n_r) = \pi(n) = w$ a reduced expression. Note that for $\ell(xs) > \ell(x)$ for $x \in W$ and $s \in S$, one has

$$xs(\Delta_+) \cap \Delta_+ \subset x(\Delta_+) \cap \Delta_+ .$$

So by applying $(*)$ inductively, we see that

$$B(n_1, \dots, n_r) = T \cdot \mathcal{U}_{w^{-1}(\Delta_+) \cap \Delta_+}$$

only depends on $w = \pi(n)$.

By 2.6. Lemma, the T-action on $\mathfrak{g}$ extends to an action of N. For $n' \in T\tilde{s}_\alpha$ with $\alpha \in \Pi$ and Θ bracket closed we get a commutative diagram

$$\begin{tikzcd} \hat{\mathfrak{n}}_\Theta \arrow[r, "n'"] \arrow[d, "\exp"'] & \hat{\mathfrak{n}}_{s_\alpha(\Theta)} \arrow[d] \\ \mathcal{U}_\Theta \arrow[r, "\gamma(n')"] & \mathcal{U}_{s_\alpha(\Theta)} \end{tikzcd}$$

where $j(n')$ is conjugation in Q_α.

Now $j(n_1, \dots, n_r)$ corresponds to the map

$n = n_1 \cdots n_r$ on $\hat{n}$ which does not depend on the expression.

(7) For $w \in W$ and $s \in S$ with $\ell(ws) > \ell(w)$, $B_w \cdot B_s = B$:

First, recall from (6), that for $x \in W$

$$B_x = T \cdot U_{x^{-1}\Delta_+ \cap \Delta_+}, B.$$

Now observe that

$$s\Delta_+ \cap \Delta_+ \cup w^{-1}\Delta_+ \cap \Delta_+ = \Delta_+$$

since $s\Delta_+ \cap \Delta_+ = \Delta_+ \setminus \{\alpha_s\}$ and $w^{-1}(\alpha_s) > 0$ since

$ws > w$, so $\alpha_s \in w^{-1}\Delta_+ \cap \Delta_+$.

Now, as in 2.7 Thm $\mathcal{U}_{w^{-1}\Delta^+\cap\Delta^+}$ and $\mathcal{U}_{s\Delta^+\cap\Delta^+}$ generate $\mathcal{U}_{\Delta_+} = \mathcal{U}$, so $B_w B_s = B$.

(8) Let $s, t \in S$, $w \in W$, s.t., $sw = wt$, $\ell(sw) > \ell(w)$. Then for any $m \in \pi^{-1}(s)$, $n \in \pi^{-1}(w)$, $b \in B \setminus B_t$, there is a $y \in (bB_t) \cap B$ and $y', y'' \in B_w$ such that for $m' = n^{-1} m n$ one gets:

- $(m')^{-1} y m' = y' m' y'' \in P_t$

- $m \gamma_w(y) m^{-1} = \gamma_n(y') m^{-1} \gamma_n(y'') \in P_s$:

Let $s = s_\alpha$ and $t = s_\beta$ for $\alpha, \beta \in \Pi$. Let

$b \in B \setminus B_t = T(\mathcal{U}_\beta \times \mathcal{U}_{\Delta_+ \setminus \{\beta\}}) \setminus T\,\mathcal{U}_{\Delta_+ \setminus \{\beta\}}$.

So the component $b_\beta \in U_\beta$ of b is non-zero.

Then $y = b_\beta \in bB_t$. Since $\ell(wt) > \ell(w)$,

$w^{-1}\beta > 0$ and

$$y \in U_\beta \subset U_{w^{-1}\Delta_+ \cap \Delta_+} \subset B_w,$$

so $y \in bB_t \cap B_w$. Recall that $m' = n^{-1} m^{-1} n$,

so $\pi(m') = w^{-1} s w = t$. Hence

$(m')^{-1} y m \in U_{t(\beta)} = U_{-\beta} \subset G_\beta$, so we can use the

Bruhat decomposition in G_β to write

$$(m')^{-1} y m = y' m' y'' \quad \text{for } y', y'' \in G_\beta \cap B = T U_\beta.$$

The map γ_n restricts to an isomorphism

$$\gamma_n : B_w \to B_{w^{-1}}$$

Since $wtw^{-1} = s$, $w(\beta) = \alpha$, so

$$\gamma_n (T U_\beta) \subset T U_\alpha .$$

So G_α and G_β are reductive group of s.s. rank 1 with isomorphic Borel subgroups. One can show that hence γ_n extends to an isomorphism $\gamma_n' : G_\beta \to G_\alpha$. and for $x \in N_\beta$, $\gamma_n'(x) = n x n^{-1}$.

Hence, we get

$$m \gamma_n(y) m^{-1} = \gamma_n(y') m^{-1} \gamma_n(y'')$$

(g) B is not normal in P_α:

Use $\tilde{s}_\alpha U_\alpha \tilde{s}_\alpha = U_{-\alpha} \not\subset B$. ✓ $\square$

Remark A: One may show that for $X \subset \Pi$ of finite type

$$Q_X = P_X = \biguplus_{\alpha \in \Pi} B s_\alpha B$$

In particular $Q_\alpha = P_s$ for $s_\alpha = s$ $\square$

Corollary: For any subsets $X, Y \subset \Pi$, we obtain

$$G = \biguplus_{w \in W_X \backslash W / W_Y} P_X w P_Y$$

In particular, we have the Bruhat decomposition

$$G = \biguplus_{w \in W} BwB$$ $\square$

Remark B: There are also the following decompositions

$$\mathfrak{g} = \biguplus_{n \in \mathcal{N}} U_- n U = \biguplus_{w \in W} U_- w B$$

$$= \biguplus_{w \in W/W_X} U_- w P_X$$

2.11 Kac-Moody groups – Representations

<u>Def</u>: (1) A $\mathfrak{g}$ (or $\hat{\mathfrak{g}}$) representation is called a pro-representation, if it is a pro-representation when restricted to each P_s (or $\mathrm{Lie}(P_s)$). Denote the categories by $\mathcal{M}(\mathfrak{g})$ and $\mathcal{M}(\hat{\mathfrak{g}})$.

(2) A $(\hat{\mathfrak{g}}, T)$-representation is a rep. for which the $\mathfrak{h}$-action integrates to T.

(3) Let $\mathcal{M}_T(\hat{\mathfrak{g}}) = \mathcal{M}(\hat{\mathfrak{g}}) \cap (\hat{\mathfrak{g}}, T)\text{-mod}$ $\square$

<u>Thm A</u>: (1) Differentiation yields an equivalence

$$\mathcal{M}(G) \to \mathcal{M}_T(\mathfrak{g})$$

(2) $\mathcal{M}_T(\hat{g})$ consist of all $(\hat{g}, T)$-reps. which are integrable for g and pro-reps. of $\hat{n}$. $\square$

<u>Cor</u>: For $\lambda \in P_+ \cap \mathfrak{h}^*_{\mathbb{Z}}$, $L(\lambda) \in \mathcal{M}_T(\hat{g})$ and hence integrates to $L(\lambda) \in \mathcal{M}(\mathcal{G})$. $\square$

Using the structure of $\mathcal{G}$ as an amalgamated product, one can also show that there is an adjoint rep.

$$\mathcal{G} \to \mathrm{Aut}(\hat{g}).$$

Moreover for any $g \in \mathcal{G}$, there is a rep. $V \in \mathcal{M}(\mathcal{G})$ s.t., g acts non-trivially on V

2.12 Kac-Moody-Flag varieties

We now sketch how $\mathcal{G}/B$, the <u>Kac-Moody flag variety</u>, can be equipped with a structure of an ind-variety. By the Bruhat decomposition

$$X = \mathcal{G}/B = \biguplus_{w \in W} X_w \quad , \text{ for } X_w = BwB/B .$$

Let $\lambda \in P_{++} \cap \mathfrak{h}^*_{\mathbb{Z}}$ and $v_\lambda \in L(\lambda)$ the highest weight vector. Then we obtain a map

$$\iota_v \; \mathcal{G}/B \longrightarrow \mathbb{P}(L(\lambda)) , \quad g \longmapsto [g v_\lambda] .$$

<u>Lemma</u>: Let $w = s_{i_1} \cdots s_{i_n}$ be a reduced expression. Then

$\langle B s_{i_j} B \rangle \subset \langle BwB \rangle \subset \mathcal{G}$ for all i.

Cor: The map λ is injective.

Proof: Since $\lambda \in P_{++}$, $\langle \lambda, \alpha_i^\vee \rangle > 0 \Rightarrow s_i(\lambda) \neq \lambda$.

Hence $\tilde{s}_i \notin \mathrm{Stab}_g([v_\lambda])$. Let $g \in \mathrm{Stab}_g([v_\lambda]) \cap B\dot{w}B$.

By lemma, $w = e$. $\square$

Recall that the Bruhat order on (W, S) is defined

by $x \leq y$, if there is a red. exp. $s_1 \cdots s_n = y$

and $x = s_1^{\varepsilon_1} \cdots s_n^{\varepsilon_n}$ for $\varepsilon_i \in \{0,1\}$ arises as

a subexpression.

For $w \in W$, $X_{\leq w} = \bigcup_{w' \leq w} X_{w'}$ is called a Schubert variety. To equip $X_{\leq w}$ with the structure of a variety, let $w = s_{i_1} \cdots s_{i_n}$ be a reduced expression. Write $\underline{w} = (s_{i_1}, \ldots, s_{i_n})$.

Then, there is a surjection

$$m_{\underline{w}} : \quad P_{s_{i_1}} \overset{B}{\times} P_{s_{i_2}} \overset{B}{\times} \cdots \overset{B}{\times} P_{s_{i_n}} / B \to X_{\leq w}$$

called the <u>Bott-Samelson-resolution</u>. The domain is actually a projective variety, using

$$P_s/B = G_s / B \cap G_s = \mathbb{P}^1.$$

The map $i_\lambda \circ m_{\underline{w}}$ is algebraic, so

$$X_{\leq w} \xrightarrow{1:1} \mathrm{im}(i_\lambda \circ m_{\underline{w}}) \subset \mathbb{P}(L(\lambda))$$

is a closed subset. We write $X_{\leq w}(\lambda)$ for $X_{\leq w}$ with the induced structure of a projective variety.

Now, let $X_{\leq n} = \bigcup_{\substack{w \in W \\ \ell(w) \leq n}} X_{\leq w}$ and

$$X_{\leq n}(\lambda) = \bigcup_{\substack{w \in W \\ \ell(w) \leq n}} X_{\leq w}(\lambda) \subset \mathbb{P}(L(\lambda))$$

Then one may show that $X_{\leq n}(\lambda) \hookrightarrow X_{\leq m}(\lambda)$ is a closed emedding for $n \leq m$. So we can equip $X = G/B$ with the structure of an ind-variety

$$X(\lambda) = \operatorname*{colim}_n X_{\leq n}(\lambda) = \bigcup X_{\leq n}(\lambda) \subset \mathbb{P}(L(\lambda))$$

One can relate $X_{\leq n}(\lambda)$ and $X_{\leq n}(\lambda+\mu)$ and show that $X_{\leq n}(\lambda)$ does not depend on λ if λ is big enough. This way one obtains a stable ind-variety structure

$$X = \operatorname*{colim}_n X_{\leq n}(\lambda_n)$$

for an appropriately chosen λ_n for each n.

Similar discussions work for generalized flag varieties G/P.

2.13 Affine Kac-Moody groups

Let $\mathring{G}$ be a semisimple algebraic group with Lie algebra $\mathring{\mathfrak{g}}$.

For simplicity assume that $\mathring{G}$ is simply connected.

Recall that we defined the affine Kac-Moody algebra associated to $\mathring{\mathfrak{g}}$ by

$$\mathfrak{g} = \hat{\mathcal{L}}\mathring{\mathfrak{g}} = \underbrace{\mathcal{L}\mathfrak{g}}_{\mathfrak{g}\otimes\mathbb{C}[t,t^{-1}]} \oplus \underbrace{\mathbb{C}c}_{\substack{\text{central}\\ \text{extension}}} \oplus \underbrace{\mathbb{C}d}_{\text{derivation } t\frac{d}{dt}}$$

We denote by G the associated Kac-Moody group.

Abbreviate $\mathcal{O} = \mathbb{C}[\![t]\!]$, $K = \mathbb{C}(\!(t)\!)$. Then we consider the groups

$$\mathring{G}(\mathcal{O}) \subset \mathring{G}(K)$$

called the formal (positive) loop groups.

The group $\mathbb{G}_m = \mathbb{C}^\times$ acts on K via loop rotations:

$$z \cdot P(t) = P(zt).$$

and we get an induced action on $\mathring{\mathfrak{g}}(K)$.

We may form the semidirect product

$$\mathring{\mathfrak{g}}(K) \rtimes \mathbb{G}_m$$

<u>Thm</u>: There is a short exact sequence

$$1 \to \mathbb{G}_m = \exp(\mathbb{C}c) \to \mathfrak{g} \to \mathring{\mathfrak{g}}(K) \rtimes \underset{\exp(\mathbb{C}d)}{\overset{\parallel}{\mathbb{G}_m}} \to 1 \qquad \square$$

For simplicity, we now disregard the difference between $\mathring{\mathfrak{g}}(K)$ and $\mathfrak{g}$.

Recall that $W_{aff} = \langle s_0, \mathring{W} = \langle s_1, \dots, s_\ell \rangle \rangle \cong Q^\vee \rtimes \mathring{W}$.

The map $\mathcal{O} \to \mathbb{C}$, $P(t) \mapsto P(0)$, yields a map

$$ev_{t=0} : \mathring{G}(\mathcal{O}) \to \mathring{G}(\mathbb{C}),$$

Let $I \subset \{s_1, \dots, s_n\}$ and $\mathring{P}_I \subset \mathring{G}(\mathbb{C})$ the corresponding parabolic subgroup. Then the <u>parahoric subgroup</u> P_I is defined via $P_I = ev_{t=0}^{-1}(\mathring{P}_I) \subset G(\mathcal{O})$.

For $I = \emptyset$, $P_I = B$ is called the Iwahori subgroups.

(so Iwahori = Borel for affine Kac-Moody).

For $I = \{1, \dots, \ell\}$, $P_I = G(\mathcal{O})$.

Def: $\mathring{G}(K)/B$ is called the *affine flag variety*

$Gr_{\mathring{G}} = G(K)/G(\mathcal{O})$ the *affine grassmannian* of $\mathring{G}$.

Denote by $\mathring{T} \subset \mathring{G}$ the maximal torus. For each

$\lambda \in Y(\mathring{T}) = \mathrm{Hom}(\mathbb{G}_m, \mathring{T})$, we obtain an element

$$t^\lambda \in \mathring{T}(K).$$

Then we get the Bruhat decompositions

$$G(K) = \biguplus_{(w,\lambda) \in \mathring{W} \ltimes Q^\vee} B w e^\lambda B,$$

$$= \biguplus_{\lambda \in Q^\vee} B e^\lambda G(\mathcal{O}) \quad \text{and}$$

$$= \biguplus_{\lambda \in P^\vee_+} G(\mathcal{O}) e^\lambda G(\mathcal{O})$$

These induce decompositions of the affine Grassmannian and affine flag variety.

<u>Example</u>: Let $\mathring{G} = GL_n$ (to further ease notation). Then

$$B = P_{\emptyset} = \begin{pmatrix} \mathcal{O}^\times & \mathcal{O} & \cdots & \mathcal{O} \\ t\mathcal{O} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mathcal{O} \\ t\mathcal{O} & \cdots & t\mathcal{O} & \mathcal{O}^\times \end{pmatrix}$$

and for $\lambda \in Y(T)$, $\lambda(t) = \begin{pmatrix} t^{\lambda_1} & & \\ & \ddots & \\ & & t^{\lambda_n} \end{pmatrix} = t^\lambda$.

Then $\lambda \in P_+^\vee \iff \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Then

$$GL_n(K) = \biguplus_{\substack{\lambda_1 \geq \cdots \geq \lambda_n \\ \lambda_i \in \mathbb{Z}}} GL_n(\mathcal{O})\, t^\lambda\, GL_n(\mathcal{O})$$

is simply the Smith normal form!

Final Overview

- Basic structure of Kac-Moody Lie algebras

$$\alpha_i \in \mathfrak{h}^*$$

$$A = (a_{ij}) \qquad \Delta_- \subset \mathfrak{h}^* \supset \Delta_+$$

$$a_{ij} = (\alpha_i, \alpha_j^\vee) \longleftarrow A \text{ symm.} \Rightarrow (\,,\,), r, \Omega$$

$$\tilde{\mathfrak{g}}(A)/r = \mathfrak{g}(A) = \underbrace{\mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+}_{}$$

$$f_i \in \mathfrak{n}_-, \quad \alpha_i^\vee \in \mathfrak{h}, \quad e_i \in \mathfrak{n}_+$$

$a_{ii} = 2 \Rightarrow \mathfrak{g}_{(i)} \cong \mathfrak{sl}_2$

$a_{ii} = 0 \Rightarrow$ Heisenberg

A GCM $\Rightarrow$ Serre:

$$(\operatorname{ad} e_i)^{1-a_{ij}} e_j = 0$$

$$(\operatorname{ad} f_i)^{1-a_{ij}} f_j = 0$$

- Weyl group, roots and types

Now A GCM

$$W(\Pi)$$
$$\Delta = \underset{\parallel}{\Delta_{re}} \oplus \Delta_{im}$$

$$\uparrow$$

V integrable

$$\{\lambda \in \mathfrak{h}^* \mid V_\lambda \neq 0\} = P(V) \hookleftarrow W = \langle r_i \mid r_i^2 = 1, (r_i r_j)^{m_{ij}} = 1 \rangle$$

$g(c)$ $\quad$ $a_{ij}a_{ji}$

$$\downarrow$$

$$\bigcup w(C) = X \subset \mathfrak{h}_{\mathbb{R}}$$

fund. chamber $\quad$ Tits cone

$$\operatorname{corank} A \begin{cases} 0 \to \text{finite} & A_n, B_n, \dots, G_2 & \Delta_{im} = \emptyset \\ 1 \to \text{affine} & \underbrace{A_\ell^{(1)}, \dots, E_8^{(1)}}_{\text{untwisted}}, \underbrace{A_2^{(2)}, \dots, D_4^{(3)}}_{\text{twisted}} & \Delta_{im} = \mathbb{Z}\setminus\{0\}\,\delta, \quad \delta = \sum_{i=0}^n a_i \alpha_i \\ \geq 1 \to \text{indef} & & \end{cases}$$

$$\downarrow$$

Extend X_ℓ by $\alpha_0 = -\theta = \sum_{i=1}^n a_i \alpha_i$

$$\downarrow$$

$$\mathfrak{g}(A) = \mathcal{L}(\mathring{\mathfrak{g}}) \oplus \mathbb{C}c \oplus \mathbb{C}d$$

- Rep. th.

$$\underbrace{U(n^-)\otimes\mathbb{C}_\lambda}_{\text{As } h\text{-mod}} = \overbrace{M(\lambda)}^{\substack{n_+ v^+ = 0 \\ h v^+ = \lambda(h) v^+}} \twoheadrightarrow L(\lambda) \Leftarrow \text{integrable} \Leftrightarrow \lambda \in P_+$$

$M(\lambda)$: Verma

$L(\lambda)$: simple h.w. mod

$$\Downarrow \qquad\qquad\qquad \Downarrow \ A \text{ symmetrizable}, \ \lambda \in P_+$$

$$ch(M(\lambda)) = \frac{e^\lambda}{\prod_{\alpha\in\Delta}(1-e(-\alpha))^{\operatorname{mult}\alpha}} \qquad ch(L(\lambda)) = \frac{\sum_{w\in W} e(w.\lambda)}{\prod_{\alpha\in\Delta}(1-e(-\alpha))^{\operatorname{mult}\alpha}}$$

Weyl-Kac character formula

$\Downarrow$

$A_1^{(1)} \to$ Jacobi-triple-identity,

Rogers-Ramanujan,

$\vdots$

- Kac-Moody group

$$\begin{tikzcd}[row sep=small]
 & & \hat{\mathfrak{n}}^+ \arrow[d, phantom, "\shortparallel"] & & r_i \rightsquigarrow \tilde{r}_i \\
\hat{\mathfrak{g}} = \mathfrak{n}_- \oplus & \mathfrak{h} \arrow[d] \oplus & \hat{\mathfrak{n}} \arrow[d, "\text{pro-nil} = \text{pro-uni}"] & \rightsquigarrow & W = N/T \arrow[d] \\
 & T & U & & N
\end{tikzcd}$$

$$\underbrace{T \quad U}_{B}$$

$\alpha \in \Pi$, $Q_\alpha = \underset{\substack{\uparrow \\ \text{reductive}}}{G_\alpha} \ltimes U_{\Delta^+ \setminus \{\alpha\}}$

$$G = \lim \left(\begin{tikzcd}[column sep=small]
N & & Q_\alpha & & Q_\beta & \cdots \\
 & N_\alpha \arrow[ul] \arrow[ur] & N_\beta \arrow[ull] \arrow[urr] & B \arrow[ul] \arrow[ur] & &
\end{tikzcd} \right)$$

$\rightsquigarrow (G, B, N, S)$ Tits system $\qquad G = \biguplus BwB/B$

A non-twisted affine $\qquad G \approx \mathring{G}(\mathbb{C}((t)))$