Practice Midterm 2
UCLA: Math 115A, Fall 2017

Instructor: Jens Eberhardt
Date: 08 October 2017

- This exam has 4 questions, for a total of 23 points.

- Please print your working and answers neatly.

- Write your solutions in the space provided showing working.

- Indicate your final answer clearly.

- You may write on the reverse of a page or on the blank pages found at the back of the booklet however these will not be graded unless very clearly indicated.

- Non programmable and non graphing calculators are allowed.

Name: _____

ID number: _____

Question	Points	Score
1	7	
2	5	
3	5	
4	6	
Total:	23	

1. Let $T : V \to W$ be a **one-to-one** linear transformation between two finite dimensional vector spaces V and W over a field F.

   (a) (2 points) Let $v_1, v_2, \ldots, v_r \in V$ be linearly independent. Prove that $T(v_1), T(v_2), \ldots, T(v_r)$ are linearly independent as well.

   (b) (2 points) Prove that $\dim(V) \leq \dim(W)$. *Hint:* Use the rank-nullity formula.

   (c) (2 points) Assume that $\dim(V) = \dim(W)$. Show that T is onto. *Hint:* Use the rank-nullity formula again.

   (d) (1 point) Give an example of a linear map, which is one-to-one but not onto.

Solution:

(a) Let $a_1, \ldots, a_r \in F$ such that

$$a_1 T(v_1) + \cdots + a_2 T(v_2) = 0.$$

We have to show $a_1 = \cdots = a_r = 0$. Since T is linear, we get

$$T(a_1 v_1 + \cdots + a_2 v_2) = 0.$$

Since T is one-to-one, $N(T) = \{0\}$, hence

$$a_1 v_1 + \cdots + a_2 v_2 = 0.$$

Since $v_1 \ldots, v_r \in V$ are linearly independent, we get $a_1 = \cdots = a_r = 0$. Hence $T(v_1), \ldots, T(v_r)$ is linearly independent.

(b) By the rank-nullity formula

$$\dim(V) = \operatorname{rank}(T) + \operatorname{nullity}(T).$$

Since $R(T)$ is a subspace of W, we know

$$\operatorname{rank}(T) = \dim(R(T)) \leq \dim(W).$$

Furthermore, since T is one-to-one

$$\operatorname{nullity}(T) = \dim(N(T)) = 0.$$

Putting all equations together, we get

$$\dim(V) = \operatorname{rank}(T) \leq \dim(W).$$

(c) As in the last part, we get

$$\dim(V) = \operatorname{rank}(T) \leq \dim(W).$$

By assumption we have $\dim(V) = \dim(W)$. Hence

$$\dim(R(T)) = \operatorname{rank}(T) = \dim(W).$$

Since $R(T)$ is a subspace of W, it follows that $R(T) = W$, which means that T is onto.

(d) Consider the map

$$T : \mathbb{R}^2 \to \mathbb{R}^3, \; T(x, y) = (x, y, 0).$$

Then T is clearly one-to-one. But $(0, 0, 1)$ is not in $R(T)$, hence T is not onto.

2. Let $\beta = \{1, x, x^2\}$ and $\beta' = \{1+x+x^2, x+x^2, x^2\}$ be bases of $V = P_2(\mathbb{R})$, the vector space of polynomials of degree less than or equal two.

   (a) (1 point) Determine the change of coordinate matrix from β' to β

   $$[I_V]_{\beta'}^{\beta}$$

   (b) (3 points) Determine the change of coordinate matrix from β to β'

   $$[I_V]_{\beta}^{\beta'}$$

   (c) (1 point) Let $T : V \to V$ be a linear transformation. Assume that you know $[T]_{\beta}^{\beta}$. What is $[T]_{\beta'}^{\beta'}$?

Solution:

   (a) In this case, we can read the matrix right off:

   $$[I_V]_{\beta'}^{\beta} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

   (b) We know that $[I_V]_{\beta}^{\beta'} = ([I_V]_{\beta'}^{\beta})^{-1}$. We hence have to invert $[I_V]_{\beta'}^{\beta}$.

   $$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

   Substract the first row from the second and third:

   $$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \end{array}\right]$$

   Substract the second row from the third:

   $$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -2 & -1 & 1 \end{array}\right]$$

   Hence

   $$[I_V]_{\beta}^{\beta'} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & -1 & 1 \end{bmatrix}$$

   (c) We have

   $$[T]_{\beta'}^{\beta'} = [I_V]_{\beta}^{\beta'} [T]_{\beta}^{\beta} [I_V]_{\beta'}^{\beta}$$

3. Let $T, U : V \to W$ be linear transformations between two finite dimensional vector spaces over a field F with bases β and γ, respectively.

   (a) (2 points) Prove that
   $$T + U : V \to W, \ (T + U)(v) = T(v) + U(v)$$
   is a linear transformation.

   (b) (1 point) Let $a \in F$. Assume that you know $A = [T]_\beta^\gamma$ and $B = [U]_\beta^\gamma$. Express is $[aT + U]_\beta^\gamma$ in terms of A and B.

   (c) (2 points) Let $v \in V$. Prove that
   $$Z = \{T : V \to W \,|\, T \text{ linear and } T(v) = 0\}$$
   is a subspace of
   $$\mathcal{L}(V, W) = \{T : V \to W \,|\, T \text{ linear}\}.$$

Solution:

(a) Let $v, w \in V$ and $c \in F$. Then

$$
\begin{aligned}
(T + U)(cv + w) &= T(cv + w) + U(cv + w) \\
&= cT(v) + T(w) + cU(v) + U(w) \\
&= c(T(v) + U(v)) + (T(w) + T(w)) \\
&= c(T + U)(v) + (T + U)(w),
\end{aligned}
$$

where we used the linearity of T and U in the second equation. Hence $T + U$ is linear.

(b) We know that $[-]_\beta^\gamma$ is linear. Hence

$$[aT + U]_\beta^\gamma = a[T]_\beta^\gamma + [U]_\beta^\gamma.$$

(c) Let $T_0 : V \to W$, $T_0(v) = 0$. This is the zero element of $\mathcal{L}(V, W)$. Clearly $T_0(v) = 0$. Hence $T_0 \in Z$. Now let $T, U \in Z$ and $a \in F$, then

$$(aT + U)(v) = aT(v) + U(v) = a0 + 0 = 0.$$

Hence also $aT + U \in Z$ and Z is a subspace.

4. Let $V = P_3(\mathbb{R})$ and $W = M_{2,2}(\mathbb{R})$. Let

$$\beta = \{1, x, x^2, x^3\}$$

$$\gamma = \left\{ w_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, w_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, w_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, w_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

be the standard ordered bases. Consider the linear map $T : V \to$ defined by

$$T(ax^3 + bx^2 + cx^1 + d) = \begin{bmatrix} a+b & c+d \\ a+c & b+c \end{bmatrix}$$

(a) (2 points) Determine $A = [T]_\beta^\gamma$.

(b) (3 points) Prove that T is an isomorphism.

(c) (1 point) Prove that V and W are isomorphic without using T.

Solution:

(a) We have to express $T(1), \ldots, T(x^3)$ in the basis γ.

$$T(1) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = w_2$$

$$T(x) = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = w_2 + w_3 + w_4$$

$$T(x^2) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = w_1 + w_4$$

$$T(x^3) = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = w_1 + w_3$$

Hence

$$A = [T]_\beta^\gamma = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}.$$

(b) We know that T is an isomorphism if and only of A is invertible. To show that A is invertible, we compute its determinant by evaluating along the first column. So $\det(A) = -\det(B)$ where

$$B = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

We compute $\det B = 2$ with our favorite technique. Hence $\det(A) = -2 \neq 0$. Hence A is invertible and T is an isomorphism.

(c) This follows from $\dim V = 4 = \dim W$.

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