Practice Midterm 1

UCLA: Math 115A, Fall 2017

Instructor: Jens Eberhardt
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- This exam has 4 questions, for a total of 16 points.
- Please print your working and answers neatly.
- Write your solutions in the space provided showing working.
- Indicate your final answer clearly.
- You may write on the reverse of a page or on the blank pages found at the back of the booklet however these will not be graded unless very clearly indicated.
- Non programmable and non graphing calculators are allowed.

Name: ______________________________

ID number: ______________________________

Question	Points	Score
1	4	
2	4	
3	4	
4	4	
Total:	16	

1. Prove or disprove that the following subsets W of the $\mathbb{R}$-vector space $V = \mathbb{R}^3$ are subspaces.

   (a) (2 points)
$$W = \{(a, b, c) \in \mathbb{R}^3 \mid a^2 + b^2 + c^2 = 0\}$$

   (b) (2 points)
$$W = \{(a, b, c) \in \mathbb{R}^3 \mid a + b + c = 0\}$$

Solution:

(a) Let $a, b, c \in \mathbb{R}$ with $a^2 + b^2 + c^2 = 0$. Since a^2, b^2, and c^2 are all bigger or equal than zero, it follows that $a = b = c = 0$. Hence $W = \{0\}$, which is clearly a subspace.

(b) Clearly the zero vector $(0, 0, 0)$ is in W. Now let (a, b, c), $(a', b', c') \in W$. Then

$$a + b + c = 0 \text{ and } a' + b' + c' = 0 \text{ implies}$$
$$a + a' + b + b' + c + c' = a + b + c + a' + b' + c' = 0.$$

So $(a, b, c) + (a', b', c') = (a + a', b + b', c + c') \in W$. Let $\lambda \in \mathbb{R}$. Then

$$a + b + c = 0 \text{ implies}$$
$$\lambda a + \lambda b + \lambda c = \lambda(a + b + c) = 0.$$

Hence $\lambda(a, b, c) = (\lambda a, \lambda b, \lambda c) \in W$. We conclude that W is a subspace of V.

2. (4 points) Let $S = \{(1,-1,0),(0,1,-1),(1,1,1)\} \subseteq \mathbb{R}^3$. Prove or disprove that S is a basis of $\mathbb{R}^3$

Solution: Since the dimension of $\mathbb{R}^3$ is 3 and S has exactly 3 elements, it suffices to prove or disprove that S is linearly independent. Let $a, b, c \in \mathbb{R}$ such that

$$a(1,-1,0) + b(0,1,-1) + c(1,1,1) = (a+c, -a+b+c, -b+c) = (0,0,0).$$

This amounts to the following system of linear equations:

$$\begin{aligned} a + c &= 0 \\ -a + b + c &= 0 \\ -b + c &= 0 \end{aligned}$$

Add (I) to (II):

$$\begin{aligned} a + c &= 0 \\ b + 2c &= 0 \\ -b + c &= 0 \end{aligned}$$

Add (II) to (III):

$$\begin{aligned} a + c &= 0 \\ b + 2c &= 0 \\ 3c &= 0 \end{aligned}$$

So $c = 0$ and also $a, b = 0$. Hence S is linearly independent and a basis of $\mathbb{R}^3$.

3. Let W_1, W_2 be subspaces of a vector space V over a field F. Prove or disprove that the following subsets are also subspaces of V.

   (a) (2 points) The intersection of W_1 and W_2

   $$W_1 \cap W_2 = \{v \in V \mid v \in W_1 \text{ and } v \in W_2\}.$$

   .

   (b) (2 points) The difference of W_1 and W_2

   $$W_1 \backslash W_2 = \{v \in V \mid v \in W_1 \text{ and } v \text{ is not an element of } W_2\}.$$

   .

Solution:

(a) Since W_1 and W_2 are subspaces of V we have $0 \in W_1$ and $0 \in W_2$. Hence $0 \in W_1 \cap W_2$.
Let $x, y \in W_1 \cap W_2$. Then $x, y \in W_1$ and $x, y \in W_2$. Since W_1 and W_2 are subspaces, $x + y \in W_1$ and $x + y \in W_2$. Hence $x + y \in W_1 \cap W_2$.
Now let $\lambda \in F$. Since W_1 and W_2 are subspaces, $\lambda x \in W_1$ and $\lambda x \in W_2$. Hence $\lambda x \in W_1 \cap W_2$.
We conclude that $W_1 \cap W_2$ is a subspace of V.

(b) Since W_2 is a subspace of V, we have $0 \in W_2$. Hence 0 is not in $W_1 \backslash W_2$, which is hence not a subspace.

4. Let V be a vector space over a field F and let $x, y, z \in V$. Prove the each of following statements or disprove them providing a counterexample.

   (a) (2 points) Assume that

$$z \in \operatorname{Span}(x, y) \text{ and}$$
$$x \in \operatorname{Span}(y, z).$$

   Then also $y \in \operatorname{Span}(x, z)$.

   (b) (2 points) Assume that

$$x \neq 0 \text{ and}$$
$$x \in \operatorname{Span}(y).$$

   Then also $y \in \operatorname{Span}(x)$.

Solution:

(a) We provide a counterexample. Let $F = \mathbb{R}$, $V = \mathbb{R}^2$ and $x, z = (0, 0)$ and $y = (1, 1)$. Then $z \in \operatorname{Span}(x, y)$ and $x \in \operatorname{Span}(y, z)$ since

$$0x + 0y = (0, 0, 0) = z \text{ and}$$
$$0y + 0z = (0, 0, 0) = x.$$

But y is not in $\operatorname{Span}(x, y) = \{0\}$.

(b) We prove the statement. Since $x \in \operatorname{Span}(y)$, there is a $\lambda \in F$ such that

$$x = \lambda y.$$

Since $x \neq 0$ clearly also $\lambda \neq 0$. Hence

$$y = \lambda^{-1} x$$

and $y \in \operatorname{Span}(x)$.

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