Motives in Geometric Representation Theory

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I. Introduction

Idea:

$$\{\text{representations of } G\} \overset{1:1}{\longleftrightarrow} \{\text{geometric objects on } X\}$$

Examples

= vector bundle + flat connection

$\{\text{rep's of } \mathbb{Z}\} \xleftrightarrow{\mathbb{Z} = \pi_1(S^1)} \{$ local systems on $S^1 = \bigcirc \}$

$\{\text{algebraic rep's of } \mathbb{C}^*\} \xleftrightarrow{\mathbb{Z} = \mathrm{Hom}_{alg}(\mathbb{C}^*, \mathbb{C}^*)} \{$ local systems (with finite support) on $\mathbb{Z} = \dots \cdot \cdot \cdot \dots \}$

$\{\text{algebraic rep's of } G/k \text{ reductive}\} \xleftrightarrow{\text{geometric Satake}} \{$ perverse sheaves (equivariant) on $Gr_{G^\vee} =$ affine grassmannian $\}$

$\{\text{rep's of } U(\mathfrak{g})/Z(\mathfrak{g})\} \xleftrightarrow{\text{Localisation}} \{$ D-modules on G/B = flag variety $\}$

This talk

motivic sheaves

II. Motivic sheaves

Idea:

motivic sheaves are to motivic cohomology
what constructible sheaves are to singular cohomology

Motivic Cohomology: Beilinson '87

$$\underset{\substack{\uparrow\\ \text{motivic cohomology}}}{H^{i,j}_{\mathcal{M}}(X)} \overset{\text{def}}{=} \underset{\substack{\uparrow\\ \text{Bloch's higher Chow groups}}}{CH^{p}(X,q)} \underset{\substack{\uparrow\\ \Leftarrow \text{std. conjectures}}}{\overset{?}{=}} \mathrm{Ext}^{i}_{\underset{\substack{\uparrow\\ \text{abelian category of mixed motives}}}{\mathcal{MM}}}(M(X), \mathbb{Z}(j))$$

Voevodsky '96

$$H^{i,j}_{\mathcal{M}}(X) = \mathrm{Hom}_{\underset{\substack{\uparrow\\ \text{triangulated cat. of mixed motives}}}{DM}}(M(X), \mathbb{Z}[i](j))$$

Motivic sheaves: Ayoub '07, Cisinski Déglise '11 :

A formalism of motivic sheaves is a system of $\otimes$-triangulated cat.

$$X \longmapsto \mathcal{H}(X), \text{ s.t.}$$

- Functoriality: $f : X \to Y \rightsquigarrow f^*, f_*, f_!, f^!$
- $\mathbb{A}^1$-invariance: $p_* p^* = id$, $p : \mathbb{A}^n_X \to X$
- $\mathbb{P}^1$-stability: $p_* p^* \mathbb{1}_X = \mathbb{1}_X \oplus \mathbb{1}_X(1)[2]$, $p : \mathbb{P}^1_X \to X$

  "Tate twist" $= \mathbb{1}_X(1)$ is $\otimes$-invertible
- localization: $X = U \sqcup Z \Rightarrow i_* i^* \to id \to j_! j^! \xrightarrow{+1}$
- base change, projection formula, etc. ...

$$\rightsquigarrow \quad H^{i,j}_{\mathcal{H}}(X) \overset{def}{:=} \mathrm{Hom}_{\mathcal{H}}(\mathbb{1}, \mathbb{1}[i](j))$$

is a generalized motivic cohomology theory

Examples:

$\mathcal{H}$	X/k	Coefficients	$H^{ij}_{\mathcal{H}}$	equivariant?
DM = Beilinson motives [CD]	$k = \mathbb{Z}$	$\mathbb{Q}$	higher Chow groups	✓ SW
DK = K-motives [CD]	$k = \mathbb{Z}$	$\mathbb{Q}$	(homotopy inv.) K-theory	✓ Hoyois
DMK = Milnor-K-motives [E.-Kelly]	$k = \mathbb{F}_p$	$\mathbb{F}_p$	higher Chow groups	(✓) [E.-Kelly in preparation]

$\vdots$

III Mixed Tate motives:

Mixed Tate motives $\approx$ motives of spaces with cell decomposition

$$\mathcal{H}^{MT} := \langle \mathbb{1}(n) \mid n \in \mathbb{Z} \rangle_{\Delta}$$

$$DM^{MT}(\mathbb{A}^n/\mathbb{F}_p, \mathbb{Q}) \cong D^b(\mathbb{Q}\text{-mod}^{\mathbb{Z}})$$

$$DK^{MT}(\mathbb{A}^n/\mathbb{F}_p, \mathbb{Q}) \cong D^b(\mathbb{Q}\text{-mod})$$

$$DMK^{MT}(\mathbb{A}^n/\mathbb{F}_p, \mathbb{F}_p) \simeq D^b(\mathbb{F}_p\text{-mod}^{\mathbb{Z}})$$

Stratified mixed Tate motives

$$X = \bigcup_{s \in \mathcal{S}} X_s \quad \text{stratification:}$$

$$\mathcal{H}^{MT}_{\mathcal{S}}(X) := \{ M \in \mathcal{H}(X) \mid M|_{X_s} \in \mathcal{H}^{MT}(X_s) \}$$

IV Applications in Rep Th:

1. BGG cat. $\mathcal{O}$

$\mathfrak{g}/\mathbb{C}$ complex reductive Lie algebra $\rightsquigarrow$ $\mathcal{O}(\mathfrak{g}) \subset \operatorname{Rep}(\mathfrak{g})$

"generated by highest weight modules"

$X^\vee = G^\vee / B^\vee$ = Langlands dual flag var.

Thm Soergel '90

$$\operatorname{proj} \mathcal{O}_0^{\mathbb{Z}}(\mathfrak{g}) \cong \langle IC(\bar{X}_w^\vee) \mid w \in W \rangle_{[n],\oplus} \subset D(X^\vee(\mathbb{C}), \mathbb{C})$$

Thm Soergel Wendt '14

$$D^b(\mathcal{O}_0^{\mathbb{Z}}(\mathfrak{g})) \cong DM_{(B^\vee)}^{MT}(X^\vee/\mathbb{F}_p, \mathbb{C})$$

2. Modular Cat. $\mathcal{O}$: $G/\mathbb{F}_p$ reductive alg. group

$\mathrm{Irr}\,\mathrm{Rep}(G) \overset{1:1}{\longleftrightarrow}$ pos. dominant weights

Soergel '01

$\mathcal{O}(G) =$ subquotient of $\mathrm{Rep}(G)$, s.t.,

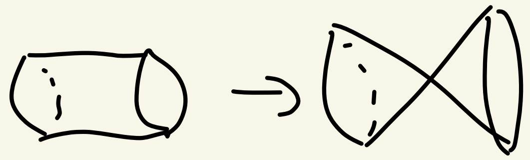
$\mathrm{Irr}\,\mathrm{Rep}\,\mathcal{O}(G) \overset{1:1}{\longleftrightarrow} W$ = Weyl group

Thm [F.] - Kelly '17

$$D^b(\mathcal{O}^{\mathbb{Z}}(G)) \xleftrightarrow{\sim} DMK^{MT}_{(B)}(X^\vee/\mathbb{F}_p, \mathbb{F}_p)$$

3 Springer theory

$\mathcal{N} \subset \mathfrak{g}$ nilpotent cone

$\mu : \tilde{\mathcal{N}} \to \mathcal{N}$ Springer resolution 

$Z = \tilde{\mathcal{N}} \times_{\mathcal{N}} \tilde{\mathcal{N}}$ Steinberg var.

Springer, Lusztig, Ginzburg:

$$H^{BM}_{top}(Z, \mathbb{Q}) = \mathbb{Q}W, \qquad CH^{\bullet}_{G \times \mathbb{G}_m}(Z, \mathbb{Q}) = \underset{\substack{\uparrow \\ \text{graded affine Hecke}}}{\mathbb{H}^{\bullet}}$$

<u>Thm E'18</u>.

- $M(\underset{\substack{\uparrow \\ \text{Springer fibre}}}{\mu^{-1}(x)})$ is pure Tate, i.e. $\bigoplus \mathbb{1}(n)[2n]$'s

- $DM^{Spr}(\mathcal{N}, \mathbb{Q}) \overset{def}{=} \langle \mu_*(\mathbb{1}_{\tilde{\mathcal{N}}}) \rangle_{\Delta} \cong D^b(\mathbb{H}^{\bullet}\text{-mod}^{\mathbb{Z}})$

Future: KLR-algebras, ...

4. K-Theory + Koszul duality

Thm E.'19

$$\begin{tikzcd}
(n) \arrow[r, leftrightarrow] & (n)[2n] \\
DM^{MT}_{(B)}(X, \mathbb{Q}) \arrow[r, leftrightarrow, "\sim", "BGS'96 + SW'14"'] \arrow[d, "\text{``forgets } (n)\text{''}"'] & DM^{MT}_{(B^\vee)}(X^\vee, \mathbb{Q}) \arrow[d, "\text{``forgets } (n)[2n]\text{''}"] \\
D^b_{(B)}(X(\mathbb{C}), \mathbb{Q}) \arrow[r, leftrightarrow, "\sim"] & DK^{MT}_{(B^\vee)}(X^\vee, \mathbb{Q})
\end{tikzcd}$$

$\rightsquigarrow$ "K-motives are Koszul-dual to constructible sheaves"

Other applications

- Richarz-Scholbach: motivic Satake
- Independence of ℓ results!
- ...