A Formalism of Mixed Sheaves in Positive Characteristic

Albert-Ludwigs-Universität Freiburg

Jens Niklas Eberhardt

25. Januar 2017

Joint work with **Shane Kelly**.

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Mixed sheaves on X
(mixed ℓ-adic sheaves, mixed Hodge modules, ...)

$$\downarrow \text{realization}$$

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Frobenius acting on $\mathrm{H}^i_{ét}(X/\overline{\mathbb{F}}_p, \mathbb{Z}/\ell)$

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- *a full six functor formalism (using Ayoub, Cisinski–Déglise),*
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- *and computes higher Chow groups*

  $$\mathrm{CH}^n(X, 2n-i; \Bbbk) \cong \mathrm{Hom}_{\mathsf{H}(X,\Bbbk)}(\mathbb{1}_X, \mathbb{1}_X(n)[i])$$

  for $X/\overline{\mathbb{F}}_p$ smooth (using Geisser-Levine).

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Shadow of graded Finkelberg-Mirkovic conjecture.