# GK-SEMINAR SS2015: SHEAF COHOMOLOGY

FLORIAN BECK, JENS EBERHARDT, NATALIE PETERNELL

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## 1. Introduction

In this seminar we want to learn how to interpret different notions of cohomology of a topological space $X$ as derived functors via the abelian category of sheaves on $X$. Many results known from algebraic topology, like the universal coefficient theorem, the Künneth formula, Poincaré duality and the Leray spectral sequence, can be generalized to families of topological spaces by Grothendieck's formalism of six functors $f^*, Rf_*, Rf_!, f^!, \otimes^L, R\mathscr{H}om$ between the derived categories of sheaves on two spaces $X$ and $Y$ for a sufficiently nice map $f : X \to Y$.

## 2. Talks

2.1. **Introduction: Jordan curve theorem.** (22.04., Wolfgang Soergel) The aim of our first talk is to illustrate some of the methods that we want to learn in this seminar by a simple example. We choose to prove the well-known Jordan curve theorem to see sheaf cohomology with compact support, derived functors, distinguished triangles and an instance of Verdier duality in action. All these concepts will be studied in detail in the following talks.

2.2. **Derived categories.** (29.04.) Derived categories give the right framework to define derived functors. Rather than going into technical details this talk is meant to give some intuition and enable us to properly discuss derived functors in the subsequent talks.

Motivate derived categories:
- Good framework for derived functors (compare second talk below);
- Explain the 'Philosophy': Complexes are *good*, cohomology is *bad*, cf. [9].

Define derived categories via their universal property (please, clearly distinguish between the definition and the possible construction). Example: Cohomology factors through the derived category.

Show the construction of the derived category of an abelian category $\mathcal{A}$:
- Shortly introduce the idea of localizing classes of morphisms and Ore localization to get the unbounded derived category $D(\mathcal{A})$.(Problem here: Hard to see that the localized category is additive.)
- Focus mainly on the construction of the bounded below derived category $D^+(\mathcal{A})$ via the bounded below homotopy category $K^+(\mathcal{I})$ of injective objects $\mathcal{I}$. (Only mention that the same construction is possible with projective objects.)

The derived category fails to be abelian: Motivate the introduction of distinguished triangles as a replacement for short exact sequences.

Introduce mapping cones and cylinders. Compare this to the corresponding topological notions (cf. [9]).

Define the family of distinguished triangles in the derived category and deduce the long exact sequence in cohomology.

Optional: Mention without proof that the derived category is a triangulated category.

*References*. Derived category: [4], III.2.-3., [11], 10.1-10.4. Distinguished triangles: [4], III.3. [11], 1.5, 10.2.

2.3. **Derived functors.** (06.05., Natalie) A left (not necessarily right) exact additive functor $F : \mathcal{A} \to \mathcal{B}$ between two abelian categories induces the derived functor $RF : D(\mathcal{A}) \to D(\mathcal{B})$ between their derived categories.

Motivate derived functors:
- $RF$ preserves distinguished triangles and can therefore be seen as an 'exact version' of $F$. Applying cohomology, this yields that a short exact sequence in $\mathcal{A}$ induces a long exact cohomology sequence in $\mathcal{B}$ in terms of the classical derived functors $R^iF := H^iRF$.
- Discuss the example of $\mathrm{Hom}_{\mathbb{Z}}(-,\mathbb{Z})$ which fails to be a duality, e.g. $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z},\mathbb{Z}) = 0$. This is fixed by the derived functor $R\mathrm{Hom}_{\mathbb{Z}}(-,\mathbb{Z})$ that turns out to be a duality on $D^b(\mathbb{Z}\text{-}mod^{fg})$.

Define $RF$ by the universal property (please, clearly distinguish between the definition and the possible construction).

Introduce $F$-acyclic objects. Show that injective (projective) objects are always $F$-acyclic.

Show how to construct the derived functor $RF$ by using resolutions with acyclic objects.

Example: Refer back to $R\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z},\mathbb{Z})$.

Give different definitions of $\mathrm{Ext}^\bullet$ via: $R^\bullet\mathrm{Hom}$ and $\mathrm{Hom}_{D^+(\mathcal{A})}$ and if you still have time showcase Yoneda's construction. In any case, explain the ring structure on $\mathrm{Ext}^\bullet(A, A)$. This interpretation for $\mathrm{Ext}^\bullet$ explains how $D(\mathcal{A})$ encodes how far $\mathcal{A}$ is from being semi-simple.

*References*. Derived functors: [4], III.6., [11], 10.5. Different definitions of Ext: [4], III.5., III.6.14., [11], 3.5.

2.4. **Sheaf cohomology.** (13.05., Max) We want to interpret cohomology of a space $X$ as derived functors and express facts about cohomology in terms of relations among these functors. To talk about derived functors, we need an abelian category to start with. We take the abelian category $Sh(X)$ of sheaves (of abelian groups) on $X$.

Recall that a continuous map $f : X \to Y$ induces the pair of adjoint functors $(f^*, f_*)$ between the categories of sheaves on $X$ and $Y$. Introduce $f_!$ and only mention that we will also obtain a pair of adjoint functors $(Rf_!, f^!)$ in the derived category in a later talk.
Show how to get the notions of global (compactly supported) sections, constant and skyscraper sheaf as well as stalk from these functors via the maps $X \to \{pt\}$ and $\{pt\} \to X$. Mention that $f^*$ is exact whereas $f_*$ and $f_!$ are in general only left exact.
Introduce $Rf_*$ ($Rf_!$), in particular $R\Gamma_{(c)}(X, -)$, by mentioning that $Sh(X)$ has enough injectives (Godement resolution), and define sheaf cohomology (with compact support). Show the different ways to get the cohomology (with compact support) of the constant sheaf $\underline{\mathbb{Z}}_X$ via: $R^\bullet\Gamma_{(c)}(X, \underline{\mathbb{Z}}_X) = R^\bullet a_{*(!)}a^*(\mathbb{Z})$ (for $a : X \to \{pt\}$) and $R^\bullet\mathrm{Hom}(\underline{\mathbb{Z}}_X, \underline{\mathbb{Z}}_X) = \mathrm{Ext}^\bullet(\underline{\mathbb{Z}}_X, \underline{\mathbb{Z}}_X)$ which naturally yields a ring structure on cohomology.
Mention the underlying presheaf of $Rf_*\mathcal{F}$ and the stalks of $Rf_!\mathcal{F}$.
Show how the unit $\mathcal{F} \to f_*f^*\mathcal{F}$ of the adjunction $(f^*, f_*)$ induces a map $H^\bullet(Y, \mathcal{F}) \to H^\bullet(X, f^*\mathcal{F})$. This yields a pullback in cohomology for $\mathcal{F} = \underline{\mathbb{Z}}_X$.

*References*. General references for sheaf theory: [5], [6], [1]. Sheaf cohomology: [4], III.8, [6], 2.6.

2.5. **de Rham's theorem.** (20.05., Anja, Felix) In certain cases sheaf cohomology can be compared with singular and de Rham cohomology. This can be done by constructing sheaf versions of the singular cochain and de Rham complex.

Discuss the different families of sheaves which are acyclic for $f_*$ ($f_!$): injective $\Rightarrow$ flabby/flasque $\Rightarrow$ soft $\Rightarrow$ c-soft. Explain to which classes continuous functions, analytic functions, compactly supported functions etc. belong.
The class of (c-)soft sheaves is acyclic for $\Gamma(X, -)$, $\Gamma_c(X, -)$ for paracompact $X$. Define the de Rham complex (of sheaves) on a smooth manifold $X$. This is a soft resolution of the constant sheaf $\underline{\mathbb{R}}_X$. We immediately get that $H^\bullet_{dR}(X, \mathbb{R}) = H^\bullet(X, \underline{\mathbb{R}}_X)$ and analogously for cohomology with compact support.
The singular cochain complex (of sheaves) on $X$ also gives a resolution of the constant sheaf. Use this to prove de Rham's theorem comparing de Rham and singular cohomology (and also with compact support).
Discuss that the ring structures on $H^\bullet(X, \underline{\mathbb{R}}_X)$ and $H^\bullet_{dR}(X, \mathbb{R})$ coincide.

*References*. Singular cohomology in the sheaf setting: [8], 17.2.10, [1], I.7, III.1. De Rham: [8], 17.2.11, [1], III.3 (notation is quite heavy here, just forget about the family of supports and different coefficients).

2.6. **Base change.** (03.06., Daniel) A cartesian diagram of locally compact spaces

$$\begin{tikzcd} W \arrow[r, "q"] \arrow[d, "g"'] & Y \arrow[d, "f"] \\ Z \arrow[r, "p"] & X \end{tikzcd}$$

induces a further relation, called proper base change, between proper direct image and inverse image. Namely, one has an isomorphism of functors

$$p^* f_! \simeq g_! q^*. \tag{2.1}$$

From this equivalence and its derived version one can deduce the projection formula, the Künneth formula and homotopy invariance of sheaf cohomology.

Prove proper base change (2.1) and deduce its derived version. If you want, you can mention here that this works in more generality, cf. [7].
Define the tensor product of sheaves and deduce the projection formula from the base change.
Discuss flat sheaves and the derived version of the projection formula. From this deduce the Künneth formula.
You can prove homotopy invariance of sheaf cohomology (meaning that two homotopic maps induce the same map in cohomology (of locally constant sheaves), cf. fourth talk) either directly or by applying the Künneth formula.

*References*. Proper base change: [8], 18.3.13, [7], [6], II.5.11, II.6.7. Homotopy invariance via base change: [8], 18.2.3, [6], II.7. Künneth formula: [5], VII.2, [6] Exercise II.18.

2.7. **Further topics: Distinguished triangles and quiver representations.** (10.06., Helene) In the first half of the talk, we discuss a distinguished triangle in the derived category $D^+(Sh(X))$ of sheaves on $X$ that is constructed out of a closed subset $i : A \hookrightarrow X$ and its complement $j : U := X \backslash A \hookrightarrow X$. This yields an interesting long exact sequence (Gysin sequence) in compactly supported cohomology, cf. first talk.
In the second half of the talk we will see how the category of constructible sheaves on simplicial complex $\mathcal{K}$ (with the natural stratification) is related to the category of representations of a certain quiver associated to $\mathcal{K}$.

*Part I:* For a locally closed subset $h : W \hookrightarrow X$ define the exceptional direct image functor $h^!$ which is a right adjoint to $h_!$ (note that $h^! = h^*$ for $W$ open and $h_! = h_*$ for $W$ closed).
Introduce the short exact sequence $0 \to j_! j^* \mathcal{F} \to \mathcal{F} \to i_* i^* \mathcal{F} \to 0$ and discuss the induced distinguished triangle in the derived category $D^+(Sh(X))$.

*Part II:* The simplicial complex $\mathcal{K}$ naturally gives rise to a quiver $Q(\mathcal{K}) := (\mathcal{K}, \subset)$. This quiver becomes a topological space via the final topology and we get a (continuous) projection map

$$p : |\mathcal{K}| \to Q(\mathcal{K}).$$

Define the category $Sh_\Delta(|\mathcal{K}|)$ consisting of sheaves on the geometric realization $|\mathcal{K}|$ of the simplicial complex $\mathcal{K}$ that are constant on simplices.
Show that $p_* : Sh_\Delta(|\mathcal{K}|) \to Sh(Q(\mathcal{K}))$ is an equivalence of categories. Finally discuss that $Sh(Q(\mathcal{K}))$ is equivalent to the category of representations of $Q(\mathcal{K})$ with additional relations (namely that two paths in $Q(\mathcal{K})$ with the same start and end point are identified). In particular, we can express cohomology of such sheaves as $Ext$-groups of quiver representations.

*References.* Distinguished triangles: [5], II.6, [8], 18.3.2, resp. [6]. For quiver representations ask us or Prof. Soergel.

2.8. **Spectral sequences of filtered complexes.** (17.06., Yi-Sheng) This talk is meant to provide the technical prerequisites for the next two talks.

Introduce the spectral sequence of a filtered complex. Try avoiding indices as much as possible and instead provide many pictures/diagrams. Explain degeneration of spectral sequences.
Apply this to construct the two natural spectral sequences associated to a double complex.

*References.* [2], Ch. 4., [3], 3.5., [4], III.7.1-5/8-9, [10], 8.3.

2.9. **Grothendieck spectral sequence.** (24.06., Antonio) Many spectral sequences, as the Čech and Leray spectral sequence, are instances of the Grothendieck spectral sequence for the composition of two derived functors. This is a spectral sequence $R^iG(R^jF) \Rightarrow R^{i+j}(G \circ F)$ deduced from the equality $R(G \circ F) = RG \circ RF$ in the derived category.

Introduce the Cartan-Eilenberg resolution and apply it to obtain the Grothendieck spectral sequence.
Explain how hypercohomology fits into this framework and mention the hypercohomology spectral sequence.
Recall Čech cohomology of a sheaf $\mathcal{F}$ with respect to a covering $\mathcal{U}$ of $X$.
Introduce the Čech spectral sequence $\check{H}^p(\mathcal{U}, \mathcal{H}^q(\mathcal{F})) \Rightarrow H^{p+q}(X, \mathcal{F})$. You can either do this directly using a spectral sequence of a double complex or via the Grothendieck spectral sequence associated to the composition of the functors

$$Sh(X) \xrightarrow{\iota} PSh(X) \xrightarrow{\check{H}^0_{\mathcal{U}}} Ab\,.$$

Mention that if $\mathcal{U}$ is $\mathcal{F}$-acyclic, then the spectral sequence degenerates on the second page.
Introduce the $\mathscr{H}om$- and $\mathscr{E}xt$-sheaves. Deduce the local-to-global spectral sequence which relates the cohomology of the $\mathscr{E}xt$-sheaves with the $Ext$-groups.

*References.* Grothendieck and hypercohomology: [4], III.7.7-15 Čech cohomology: [4], III.8.2., II.8. Local-to-global: [3], p.706

2.10. **Frölicher and Leray spectral sequence.** (01.07., Martin) The Frölicher spectral sequence for a compact Kähler manifolds $X$ gives a relation between the sheaf cohomology groups $H^p(X, \Omega_X^q)$ of the sheaves $\Omega_X^q$ of holomorphic differential forms on $X$ and the Hodge filtration on $H^{p+q}(X, \mathbb{C})$.

> Given a complex manifold $X$, define the sheaf $\Omega_X^p$ of holomorphic $p$-forms on $X$ and introduce the Dolbeault resolution $\Omega_X^p \to \Omega_X^{p,\bullet}$.
> Outline the Hodge filtration and construct the Frölicher spectral sequence. Finally, outline why the Frölicher spectral sequence degenerates at $E_2$ for a compact Kähler manifold $X$.

Another example of a Grothendieck spectral sequence is the Leray spectral sequence for a map $f : X \to Y$ which enables one to compute the cohomology of a sheaf $\mathcal{F}$ on $X$ by the cohomology of the higher direct image sheaves $R^j f_* \mathcal{F}$.

> Given two maps $f : X \to Y$, $g : Y \to Z$ recall that $f_*$ preserves injectives, hence $R(g \circ f)_* = Rg_* \circ Rf_*$. Now apply the Grothendieck spectral sequence to get $R^i g_* R^j f_* \mathcal{F} \Rightarrow R^{i+j}(g \circ f)_* \mathcal{F}$. The special case $Z = \{pt\}$ is called Leray spectral sequence.
> Apply the Leray spectral sequence to the following setup: Let $f : X \to Y$ be a holomorphic submersion between two compact Kähler manifolds $X$, $Y$ and $\mathcal{F} = \underline{\mathbb{C}}_X$. Then the Leray spectral sequence degenerates at $E_2$. Only show how relative hard Lefschetz implies that the differential on the $E_2$-page vanishes.

*References*. Frölicher: [10], 8.3.3. Leray: [3], p.462ff., [4], III.8. Example for Leray: [3], p.466ff., [12].

2.11. **Verdier Duality I.** (08.07., Florian, Jens) To a smooth manifold one can associate singular cohomology groups (with compact support), singular homology and Borel-Moore-homology. Cohomology and homology in the same degree are related by algebraic duality (universal coefficient theorem) and in complementary degree by Poincaré. All of these dualities can be understood as an instance of Verdier duality which in turn just states that there is a right adjoint functor $f^!$ to $Rf_!$. This is the aim of the last talks.

2.12. **Verdier Duality II.** (15.07., Florian, Jens)

2.13. **Six functor formalism.** (22.07., Fritz)

## References

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